Chapter 16

Riemannian spaces (intrinsic view)

Say we have a complete C^∞ atlas $\mathcal{A}_*^\infty \subset \mathcal{A}_*$ on an n-dimensional topological manifold M with an initial covering atlas $\mathcal{A}$. My objective here is to first introduce the **Riemannian metric tensor**. This requires the introduction of several linear spaces. First of all, we will assume familiarity with the filament space $\mathcal{L}_P M$ associated with each $P \in M$, and the differential maps

$$d\mathbf{p}_\mathbf{x} : T_\mathbf{x}\mathbb{R}^n \to \mathcal{L}_{\mathbf{p}(\mathbf{x})}M \qquad \text{and} \qquad d\xi_P : \mathcal{L}_P M \to T_{\xi(P)}\mathbb{R}^n \tag{16.1}$$

which are linear isomorphisms defined for each $\mathbf{x} \in U \subset \mathbb{R}^n$ and $P \in \mathbf{p}(U) \subset M$ for each chart/chart function pair $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$. Notice that in principle there are a lot of these linear spaces and differential maps that help us translate between tangent spaces at points $\mathbf{x} \in \mathbb{R}^n$ and linear filament spaces at points $P \in M$. The differential maps in (16.1) will be inverses of each other when $P = \mathbf{p}(\mathbf{x})$ or equivalently $\mathbf{x} = \xi(P)$.

A next linear space to consider is $\mathscr{T}_P^2(M)$ consisting of all **bilinear functions** $b : \mathcal{L}_P M \times \mathcal{L}_P M \to \mathbb{R}$. These are called variously

1. the bilinear **forms** at a point P,
2. the **two forms** at a point P, and
3. the **two tensors** at a point P.

All these terms are saying the same thing about a function $b : \mathcal{L}_P M \times \mathcal{L}_P M \to$

$\mathbb{R}$ which is precisely (and nothing more than) the following:

$$b(c_1 v + c_2 w, z) = c_1 b(v, z) + c_2 b(w, z) \qquad \text{and}$$
$$b(z, c_1 v + c_2 w) = c_1 b(z, v) + c_2 b(z, w)$$

for $v, w \in \mathcal{L}_P M$ and $c_1, c_2 \in \mathbb{R}$.

Exercise 16.1. Show $\mathscr{T}_P^2(M)$ is a real linear space.

Definition 20. An element $b \in \mathscr{T}_P^2 M$ is **symmetric** if

$$b(v, w) = b(w, v) \qquad \text{for} \qquad v, w \in \mathcal{L}_P M.$$

Definition 21. An element $b \in \mathscr{T}_P^2 M$ is **positive definite** if

$$b(v, v) \geq 0 \qquad \text{for} \qquad v \in \mathcal{L}_P M$$

with equality if and only if $v = \mathbf{0} \in \mathcal{L}_P M$.

We next extend the notation $\mathscr{T}_P^2(M)$ in a somewhat unorthodox[1] way. Specifically, we let $\mathscr{T}^2(M)$ denote the collection of **bilinear form fields**, that is the collection of all functions

$$b : M \to \bigcup_{Q \in M} \mathscr{T}_Q^2(M)$$

satisfying

(i) $b_P \in \mathscr{T}_P^2(M)$ where b_P denotes $b(P) \in \mathscr{T}_P^2(M)$, and

(ii) The functions $g_{ij} : U \to \mathbb{R}$ by

$$g_{ij}(\mathbf{x}) = b_{\mathbf{p}(\mathbf{x})}(d\mathbf{p}_{\mathbf{x}}(\mathbf{e}_i), d\mathbf{p}_{\mathbf{x}}(\mathbf{e}_j))$$

satisfy $g_{ij} \in C^\infty(U)$ whenever $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$.

We can say that the second condition means we are restricting attention to bilinear form fields which are **chart** C^∞. Sometimes one says explicitly that these are “chart smooth bilinear form fields” with the consideration of form fields of lower regularity possible. Form fields with lower regularity are certainly possible, but we are going to ignore them for now.

The set $\mathscr{T}^2(M)$ is referred to as the collection of

[1] and somewhat notationally irritating. . .

1. **bilinear form fields** on M,
2. **two form fields** on M, and
3. **two tensors fields** on M.

Sometimes the word "field" is left off, so an element $b \in \mathscr{T}^2(M)$ is called simply a

1. **bilinear form** on M,
2. a **two form** on M, or
3. a **two tensor** on M.

These last terms are often used to refer to the corresponding values $b_P : \mathcal{L}_P M \times \mathcal{L}_P M \to \mathbb{R}$ "at a point," without bothering to say "at a point," so one has to sort of pay attention and figure out what is intended by the context—or any way you can. For the time being, I will try to avoid getting sloppy and retain the "at a point" and/or "field," when I'm talking about forms/tensors.

The set $\mathscr{T}^2(M)$ of two tensor fields is also a real linear space.

Exercise 16.2. Find the dimension of $\mathscr{T}^2(M)$.

However, $\mathscr{T}^2(M)$ is more than a linear space:

Exercise 16.3. Show that given $b \in \mathscr{T}^2(M)$, the function

$$(fb) : M \to \bigcup_{Q \in M} \mathscr{T}^2_Q(M) \qquad \text{by} \qquad (fb)_P(v, w) = f(P)b_P(v, w)$$

is a two tensor field on M for every $f \in cC^\infty(M)$.

Exercise 16.4. Show the scaling by chart C^∞ functions defined in Exercise 16.3 satisfies the following

(a) $(fg)b = f(gb)$ for $f, g \in cC^\infty(M)$ and $b \in \mathscr{T}^2(M)$.

(b) $1b = b$ where 1 denotes the constant function on M with value 1 and $b \in \mathscr{T}^2(M)$.

(c) $(f + g)b = fb + gb$ for $f, g \in cC^\infty(M)$ and $b \in \mathscr{T}^2(M)$.

(d) $f(b_1 + b_2) = fb_1 + fb_2$ for $f \in cC^\infty(M)$ and $b_1, b_2 \in \mathscr{T}^2(M)$.

This makes $\mathscr{T}^2(M)$ a module over the ring $cC^\infty(M)$.

With these definitions/spaces we are in a position to easily and cleanly give one definition of a/the Riemannian metric tensor:

Definition 22. A **Riemannian metric tensor** on a C^∞ manifold M is a (smooth) symmetric, positive definite, two form field.

16.1 The metric in coordinates

Say we have a Riemannian metric tensor $\mu \in \mathscr{T}^2(M)$. Then given a chart/chart function $(U, \mathbf{p}) \in \mathcal{A}^\infty_*$, we can define a Riemannian metric tensor g on U (considered as a manifold) by

$$g_{\mathbf{x}}(\mathbf{v}, \mathbf{w}) = \mu_{\mathbf{p}(\mathbf{x})}(d\mathbf{p}_{\mathbf{x}}(\mathbf{v}), d\mathbf{p}_{\mathbf{x}}(\mathbf{w})).$$

Note the following:

1. In this formulation $g \in \mathscr{T}^2(U)$.
2. $g_{\mathbf{x}} \in \mathscr{T}^2_{\mathbf{x}}(U)$.
3. $g_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^n \times T_{\mathbf{x}}\mathbb{R}^n \to \mathbb{R}$.

It is natural to compare $g_{\mathbf{x}}$ to the Euclidean inner product and also to express $g_{\mathbf{x}}$ in terms of the Euclidean inner product. Specifically, given

$$\mathbf{v} = \sum_{i=1}^n v_i \mathbf{e}_i \qquad \text{and} \qquad \mathbf{w} = \sum_{j=1}^n w_j \mathbf{e}_j$$

in $T_{\mathbf{x}}U$, we have

$$\begin{aligned} g_{\mathbf{x}}(\mathbf{v}, \mathbf{w}) &= \sum_{i,j=1}^n v_i w_j g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_j) \\ &= \sum_{i=1}^n \langle \mathbf{z}_i, \mathbf{w} \rangle \\ &= \left\langle \sum_{i=1}^n \mathbf{z}_i, \mathbf{w} \right\rangle \end{aligned}$$

where

$$\mathbf{z}_i = v_i(g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_1), g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_2), \ldots, g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_n)).$$

At this point, let us be somewhat careful concerning our convention(s) for Euclidean vectors. We have (largely for typographical convenience) presented vectors $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$ as row vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. This presents certain inconviences for the usual conventions involved with linear functions on Euclidean vector spaces and the relation with matrix multiplication. Here, if we wish to be careful, we should introduce the column vector

$$\mathbf{v}^T = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

which is the transpose of $\mathbf{v}$. Then we can write

$$\mathbf{z}_i^T = v_i \begin{pmatrix} g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_1) \\ g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_2) \\ \vdots \\ g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_n) \end{pmatrix}$$

and

$$\sum_{i=1}^n \mathbf{z}_i^T = \begin{pmatrix} \sum_{i=1}^n g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_1)\ v_i \\ \sum_{i=1}^n g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_2)\ v_i \\ \vdots \\ \sum_{i=1}^n g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_n)\ v_i \end{pmatrix}.$$

Setting $g_{ij} = g_{ij}(\mathbf{x}) = g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_n)$ for $i, j = 1, 2, \ldots, n$ and denoting the matrix with g_{ij} in the i-th row and j-th column by (g_{ij}), we see this is the usual notation for the matrix multiplication $(g_{ij})\mathbf{v}^T$. That is,

$$g_{\mathbf{x}}(\mathbf{v}, \mathbf{w}) = \langle [(g_{ij})\mathbf{v}^T]^T, \mathbf{w}\rangle = \mathbf{w}(g_{ij})\mathbf{v}^T.$$

On the other hand, we also have by the symmetry $g_{ij} = g_{ji}$, so it's easy to check

$$g_{\mathbf{x}}(\mathbf{v}, \mathbf{w}) = \mathbf{v}(g_{ij})\mathbf{w}^T$$

as well.

Exercise 16.5. Use the symmetry of the metric tensor μ to show $g_{ij} = g_{ji}$ where

$$g_{\mathbf{x}}(\mathbf{v}, \mathbf{w}) = \mu_{\mathbf{p}(\mathbf{x})}(d\mathbf{p}_{\mathbf{x}}(\mathbf{v}), d\mathbf{p}_{\mathbf{x}}(\mathbf{w})) \tag{16.2}$$

and

$$g_{ij} = g_{ij}(\mathbf{x}) = g_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_j)$$

for $i, j = 1, 2, \ldots, n$.

Any bilinear form field $b \in \mathscr{T}^2(U)$ on an open subset U of $\mathbb{R}^n$ is determined by a matrix (g_{ij}) with

$$b_{\mathbf{x}}(\mathbf{v}, \mathbf{w}) = \mathbf{w}(g_{ij})\mathbf{v}^T \tag{16.3}$$

where

$$g_{ij} = g_{ij}(\mathbf{x}) = b_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_j)$$

for $i, j = 1, 2, \ldots, n$ as described above. We can do this because there are "natural" coordinates determined on U and every tangent space $T_{\mathbf{x}}U = T_{\mathbf{x}}\mathbb{R}^n$ by the standard unit basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. We need to be a little careful in this case because there are (always) two Riemannian structures on U in such a situation. One is the "usual" Riemannian structure on U in which $T_{\mathbf{x}}U = T_{\mathbf{x}}\mathbb{R}^n$ is considered with the usual Euclidean inner product. The other is where we nominally erase the Euclidean inner product (though still use use it as a technical device as in (16.3)) and consider $T_{\mathbf{x}}U$ as a linear space with the inner product determined by $b_{\mathbf{x}}$. We can even go back to filaments in $\mathcal{L}_{\mathbf{x}}U$, though that is unnecessary.

In any case, when we use the clean definition of the Riemannian metric tensor, then the function $g_{\mathbf{x}} \in \mathscr{T}^2(U)$ may be considered a coordinate expression of the metric tensor, and the corresponding matrix assignment (g_{ij}) may also be thought of as a kind of coordinate expression for the metric tensor. In particular, the functions $g_{ij} \in C^\infty(U)$ given by

$$g_{ij}(\mathbf{x}) = \mu_{\mathbf{p}(\mathbf{x})}(d\mathbf{p}_{\mathbf{x}}(\mathbf{e}_i), d\mathbf{p}_{\mathbf{x}}(\mathbf{e}_j))$$

are often called the **coordinate coefficients** of the metric tensor or more commonly **metric coefficients**.

16.1.1 Coordinate transformation formula

Once the metric is expressed in coordinates as described above, it is interesting to derive a coordinate transformation formula for the metric coefficients. Say $(V, \mathbf{q})$ is an overlapping chart/chart function so that

$$\psi = \eta \circ \mathbf{p}\Big|_{\xi(W)} \qquad \text{and} \qquad \phi = \psi^{-1} = \xi \circ \mathbf{q}\Big|_{\eta(W)}$$

are the corresponding changes of coordinates where $W = \mathbf{p}(U) \cap \mathbf{q}(V)$ as usual and also with coordinate functions $\xi = \mathbf{p}^{-1}$ and $\eta = \mathbf{q}^{-1}$ as usual. Then we have alternative metric coefficients

$$h_{ij}(\mathbf{x}) = \mu_{\mathbf{q}(\mathbf{x})}(d\mathbf{q}_{\mathbf{x}}(\mathbf{e}_i), d\mathbf{q}_{\mathbf{x}}(\mathbf{v}_j)) \tag{16.4}$$

with respect to $(V, \mathbf{q})$, and there should be a formula for (h_{ij}) in terms of (g_{ij}) and the change of variables obtained using the chain rule, that is a coordinate transformation rule (for the metric).

Starting with (16.4) we should have

$$h_{ij}(\mathbf{x}) = \mu_{\mathbf{q}(\mathbf{x})}(d\mathbf{p}_{q(\mathbf{x})} \circ d\phi_{\mathbf{x}}(\mathbf{e}_i), d\mathbf{p}_{q(\mathbf{x})} \circ d\phi_{\mathbf{x}}(\mathbf{e}_j)).$$

Exercise 16.6. Show that given overlapping chart/chart functions $(U, \mathbf{p})$ and $(V, \mathbf{q})$ and $\mathbf{v} \in T_{\mathbf{x}}V = T_{\mathbf{x}}\mathbb{R}^n$ there holds

$$d\mathbf{q}_{\mathbf{x}}(\mathbf{v}) = d\mathbf{p}_{\phi(\mathbf{x})} \circ d\phi_{\mathbf{x}}(\mathbf{v}) = d\mathbf{p}_{\phi(\mathbf{x})}([D\phi(\mathbf{x})\mathbf{v}^T]^T)$$

where $\phi = \xi \circ \mathbf{q}\Big|_{\eta(W)}$ and $W = \mathbf{p}(U) \cap \mathbf{q}(V)$ as usual.

According to (16.2) then

$$\begin{aligned} h_{ij}(\mathbf{x}) &= \mu_{\mathbf{p}\circ\phi(\mathbf{x})}(d\mathbf{p}_{\phi(\mathbf{x})}([D\phi(\mathbf{x})\mathbf{e}_i^T]^T), d\mathbf{p}_{\phi(\mathbf{x})}([D\phi(\mathbf{x})\mathbf{e}_j^T]^T) \\ &= g_{\phi(\mathbf{x})}([D\phi(\mathbf{x})\mathbf{e}_i^T]^T, [D\phi(\mathbf{x})\mathbf{e}_j^T]^T) \\ &= [D\phi(\mathbf{x})\mathbf{e}_j^T]^T \ (g_{ij}(\phi(\mathbf{x}))\ D\phi(\mathbf{x})\mathbf{e}_i^T \\ &= \mathbf{e}_j D\phi(\mathbf{x})^T \ (g_{ij}(\phi(\mathbf{x}))\ D\phi(\mathbf{x})\mathbf{e}_i^T \\ &= \mathbf{e}_j\ [D\phi(\mathbf{x})^T \ (g_{ij}(\phi(\mathbf{x}))\ D\phi(\mathbf{x})]\ \mathbf{e}_i^T. \end{aligned}$$

Recalling our convention that $\mathbf{e}_j = (0, 0, \ldots, 1, \ldots, 0)$ is a row vector with 1 in the j-th entry and zeros elsewhere, we see

$$\mathbf{e}_j\ [D\phi(\mathbf{x})^T \ (g_{ij}(\phi(\mathbf{x}))\ D\phi(\mathbf{x})]$$

is the j-th row of the matrix $D\phi(\mathbf{x})^T\ (g_{ij}(\phi(\mathbf{x}))\ D\phi(\mathbf{x})$ and

$$\mathbf{e}_j\ [D\phi(\mathbf{x})^T\ (g_{ij}(\phi(\mathbf{x}))\ D\phi(\mathbf{x})]\ \mathbf{e}_i^T$$

is the row j and column i entry of the same matrix. Thus, technically we have shown

$$(h_{ij})^T = D\phi(\mathbf{x})^T\ (g_{ij}(\phi(\mathbf{x}))\ D\phi(\mathbf{x}).$$

On the other hand, both the matrix (h_{ij}) and the matrix $D\phi(\mathbf{x})^T\ (g_{ij}(\phi(\mathbf{x}))\ D\phi(\mathbf{x})$ are symmetric, so we have also

$$(h_{ij}) = D\phi(\mathbf{x})^T\ (g_{ij}(\phi(\mathbf{x}))\ D\phi(\mathbf{x}) \tag{16.5}$$

which is the usual coordinate transformation formula for the metric coefficients under a change of coordinates.

16.1.2 An alternative formulation

The discussion of the previous section suggests an alternative to the "clean" definition of the metric tensor.

Definition 23. A **Riemannian metric tensor** on a C^∞ manifold M is a matrix assignment

$$\mu : \mathcal{A}_*^\infty \to \bigcup_{(V,\mathbf{q})\in\mathcal{A}_*^\infty} [GL_n(\mathbb{R})]^V$$

satisfying

(i) $\mu(U,\mathbf{p})$ is a C^∞ matrix valued function on U with values $(g_{ij}(\mathbf{x})) \in Sym_n^+(\mathbb{R})$ for each $\mathbf{x} \in U$ where $Sym_n^+(\mathbb{R})$ denotes the symmetric positive definite matrices in $GL_n(\mathbb{R})$.

(ii) Given overlapping chart/chart functions pairs $(U,\mathbf{p})$ and $(V,\mathbf{q})$ with the usual notation and $\mu(V,\mathbf{q})$ having values $(h_{ij}(\mathbf{x})$ for $\mathbf{x} \in U$, the coordinate transformation formula

$$(h_{ij}) = D\phi(\mathbf{x})^T\ (g_{ij}(\phi(\mathbf{x}))\ D\phi(\mathbf{x})$$

always holds (for every $(U,\mathbf{p})$ and $(V,\mathbf{q})$ and every $\mathbf{x} \in V$).

The requirement that $\mu(U, \mathbf{p}) = (g_{ij})$ is C^∞ in condition **(i)** simply means $g_{ij} \in C^\infty(U)$. Notice also that one can write

$$\mu : \mathcal{A}_*^\infty \to \bigcup_{(V,\mathbf{q})\in\mathcal{A}_*^\infty} [Sym_n^+(\mathbb{R})]^V$$

with

$$\mu(U, \mathbf{p}) \in [Sym_n^+(\mathbb{R})]^U.$$

Hopefully, it is clear why this is a less "clean" definition. Hopefully it is also more or less clear why it is an equivalent definition.

16.2 Length in M and intrinsic derivatives

Given a C^∞ Riemannian manifold, i.e., a C^∞ manifold M equipped with a Riemannian metric tensor μ as described above, we can define the length of paths. Specifically, given $\alpha \in c\mathfrak{P}^\infty(M)$ with $\alpha : [a, b] \to M$ with $a, b \in \mathbb{R}$, it is possible to find a partition $t_0 = a < t_1 < t_2 < \cdots < t_k = b$ for which there exist charts U_ℓ from chart/chart function pairs $(U_\ell, \mathbf{p}_\ell) \in \mathcal{A}_*^\infty$ for $\ell = 0, 1, 2, \ldots, k-1$ such that

$$\alpha([t_\ell, t_{\ell+1}]) \subset U_\ell \qquad \text{for} \qquad \ell = 0, 1, 2, \ldots, k-1.$$

We then define

$$\begin{aligned}\operatorname*{length}_M[\alpha] &= \sum_{\ell=0}^{k-1} \int_{t_\ell}^{t_{\ell+1}} [\mu(d(\mathbf{p}_\ell)_{\xi\circ\alpha(t)}((\xi_\ell \circ \alpha)'(t)), d(\mathbf{p}_\ell)_{\xi\circ\alpha(t)}((\xi_\ell \circ \alpha)'(t)))]^{1/2}\, dt \\ &= \sum_{\ell=0}^{k-1} \int_{t_\ell}^{t_{\ell+1}} \langle (g_{ij})(\xi_\ell \circ \alpha)'(t), (\xi_\ell \circ \alpha)'(t)\rangle^{1/2}\, dt\end{aligned}$$

where again we have lapsed into the convenient convention for matrix multiplication in which $(g_{ij})(\xi_\ell \circ \alpha)'(t)$ denotes

$$[(g_{ij})(\xi_\ell \circ \alpha)'(t)^T]^T.$$

It should be shown that this definition is independent of the partition and the particular charts used.

Near a single point only one chart is required, and this leads to the next "big" question:

What is the intrinsic derivative of a function $f \in cC^\infty(M)$?

In order to answer this question, one must return to elementary calculus. In answering this question, one is taking the first step toward adapting the concepts of elementary calculus to the Riemannian manifold M.

You may wish to consider the special case when the dimension of n is $n = 1$. Then we/you should be interested in the value of $f'(P)$ for $f : M \to \mathbb{R}$ with $f \in cC^\infty(M)$. You can certainly take a chart/chart function pair $(U, \mathbf{p})$, and in this case U can be taken as an open interval (a, b) with some $t \in (a, b)$ for which $\mathbf{p}(t) = P$. The "coordinate expression" for f is then $f \circ \mathbf{p}$, and you can certainly consider the value

$$(f \circ \mathbf{p})'(t).$$

This is not a good value for the intrinsic derivative of $f'(P)$ for $f : M \to \mathbb{R}$ however. For one thing notice that $(f \circ \mathbf{p})'(t)$ will depend on which chart/chart function you have chosen. The value of the intrinsic derivative $f'(P)$ should depend, furthermore, on the metric (and $(f \circ \mathbf{p})'(t)$ while it depends on the chart does not depend on the metric).

16.3 Orthodoxy

The more orthodox way to introduce $\mathscr{T}^2(M)$ is in terms of the module $\mathscr{X}(M)$ of (smooth) filament fields on M. You may recall that $\mathscr{X}(M)$ is a module over $cC^\infty(M)$, and therefore, one can consider bilinear forms (or perhaps more properly bimodular forms)

$$B : \mathscr{X}(M) \times \mathscr{X}(M) \to cC^\infty(M).$$

Usually $\mathscr{T}^2(M)$ denotes the functions B for which

$$\begin{aligned} B(fv + gw, z) &= fB(v, z) + gB(w, z) \qquad \text{and} \\ B(z, fv + gw) &= fB(z, v) + gB(z, w) \end{aligned}$$

where $f, g \in cC^\infty(M)$ and $v, w, z \in \mathscr{X}(M)$. In this case, one can say B is a smooth tensor field if the function $f : M \to \mathbb{R}$ given by

$$f(P) = B(v, w)_P$$

satisfies $f \in cC^\infty(M)$ for every pair of smooth filament fields $(v, w) \in \mathscr{X}(M) \times \mathscr{X}(M)$. There is of course some checking to be done to show this is an equivalent formulation for $\mathscr{T}^2(M)$.

Exercise 16.7. Using the alternative formulation for bimodular tensor fields $\mathscr{T}^2(M)$, define what is meant by a **two tensor at a point** determined by $B \in \mathscr{T}^2(M)$.