1 Philosophy: What is Geometry?

1.1 Two Theorems from Geometry

1.2 Something Different

We have been doing geometry, i.e., proving things about curves and surfaces (nice subsets) of space. This is what Euclid did when he proved the sum of the interior angles of a triangle is $180^\circ$. Now, we’re going to forget about that (i.e., forget about looking at subsets as our subject of interest) and try to generalize our notion of space. The spaces used in the last section were $\mathbb{R}^2$ and $\mathbb{R}^3$; we could follow the standard path and talk about creatures that live in two-dimensional space and generalize $\mathbb{R}^2$ to surfaces. But really, true space to our intuition is three-dimensional space; we are not bugs that live in some kind of surface. For this reason, I prefer to start with three dimensions. To set things up, we begin with a little story that should motivate the generalization. The story is a kind of science fiction, but I’ve tried not to stretch the possibilities too far.

2 Introduction to Riemannian Geometry

Episode 1: The Setup, or What They Saw

We begin with two observers in inertial observation stations in real physical space—that is our generalized space $\Sigma$—which we wish to discover.

We are assuming our observers John Oh and John Wun can determine distances from their points of observation to objects in their (three dimensional) space $\Sigma$. Furthermore, we are assuming that they can unambiguously identify all locations in $\Sigma$ denoted by points $p$.

John Oh sends John Wun out to the point $p_1$ which appears on his chart as $u_1$. From there, he predicts that John Wun will see the point $p_2$ (marked $u_2$ on John Oh’s chart) directly
to his left (90°). John Wun, however, sees $p_2$ only about 78.05° to his left. Furthermore, while John Oh predicts $p_2$ to be 78.5 thousand miles from John Wun, it is only 70.4 thousand miles as marked on John Wun’s chart.

Figure 1: John Oh’s chart of space on the left; John Wun’s chart on the right.

How can you explain these observations? What other observations would you like to ask the John’s to make so that you could explain?

2.1 Imaginary -vs- Physical; Topological Manifolds

Reflection of the observations of John Wun and John Oh suggests that their charts are missing some inherent properties of the space in which they live. There is something that it is difficult to observe directly. The space itself seems to be curving and stretching. Nevertheless, radial measurement appears to be trustworthy. We will not commence to call the space $\Sigma$ a three-dimensional Riemannian space. It will take some time to describe the essential properties of $\Sigma$, but we start with the following.

$\Sigma$ is a metric space with a distance $d$, and at each observation point $p \in \Sigma$, one (that is to say, Oh or Wun) can construct a bijective continuous map $\xi : B_r(p) \rightarrow \mathbb{R}^3$ such that $\xi(p) = 0$ and for each $q \in B_r(p)$

$$d(q, p) = |\xi(q)|.$$ (1)

We will leave aside for the moment the question of which $r$ we have in mind here. Simple considerations will soon show that (1) may only be expected for small enough $r$ in general.
There is a convenient generalization which we mention now. Namely, we may assume that given \( p \in \Sigma \), there is a bijective map \( \xi : B_r(p) \to \mathbb{R}^3 \) and some \( \mathbb{R}' \) such that

\[
d(q, q') = |\xi(q) - \xi(q')|
\]

if \( d(q, q') < r' \) and \( \xi(q) = t(\xi(q) - \xi(q')) \) for some \( t \in \mathbb{R} \). Notice that (1) becomes a special case of (2) for small enough \( r \).

**Exercise 1** Interpret geometrically the condition \( \xi(q) = t(\xi(q) - \xi(q')) \) for some \( t \in \mathbb{R} \).

We now recall a useful result from elementary topology.

**Lemma 1** If \( f : X \to Y \) is a bijective map of a compact space into a Hausdorff space, then \( f \) is a homeomorphism.

**Proof:** Exercise. See Appendix 1 for definitions.

We conclude that our postulated mapping \( \xi \) is a homeomorphism onto its image. We pause to compare our first (physically motivated) assumption on \( \Sigma \) to the definition of a topological manifold.

**Definition 1** A topological manifold is a set \( M \) with a (usually Hausdorff and second countable) topology such that for each \( p \in M \), there is a neighborhood \( U \) of \( p \), a neighborhood \( V \) of \( \mathbb{R}^n \) and a homeomorphism \( \xi : U \to V \). The mapping \( \xi \) is called a coordinate chart; its inverse is called a parameter chart.

Notice that the Riemannian space \( \Sigma \) is a topological manifold.
2.2 Overlapping Experience; Differentiable Manifolds

The experience of our observers suggests useful information is obtained by comparing observations (and paths in particular) between charts based at distinct points. A corollary is that when \( \xi : U \to \mathbb{R}^3 \) and \( \tilde{\xi} : \tilde{U} \to \tilde{V} \) are two such charts, one should study the maps \( \xi \circ \xi^{-1} \) and \( \tilde{\xi} \circ \tilde{\xi}^{-1} \) on the intersection domains \( \xi(U \cap \tilde{U}) \) and \( \tilde{\xi}(U \cap \tilde{U}) \).

Observations suggest, furthermore, that if \( \gamma : [0, d_1] \to \xi(U \cap \tilde{U}) \) parameterizes a smooth curve, then

\[
\gamma = \xi \circ \xi^{-1} \circ \tilde{\gamma}
\]

parameterizes a smooth curve in \( \xi(U \cap \tilde{U}) \), though the path itself may be stretched. This is a way of saying \( \Sigma \) itself is smooth; nothing happens to the curve in passing through \( \Sigma \) to destroy its smoothness in the chart.

**Lemma 2** If \( f : B_r(x_0) \to \phi(B_r(x_0)) \) is a homeomorphism of neighborhoods of \( \mathbb{R}^n \) and

1. \( f^{-1} \circ \tilde{\gamma} \) parameterizes a smooth curve whenever \( \tilde{\gamma} \) parameterizes a smooth curve in \( f(B_r(x_0)) \) and

2. \( f \circ \gamma \) parameterizes a smooth curve whenever \( \gamma \) parameterizes a smooth path in \( B_r(x_0) \),

then \( f \) is smooth and \( df_{x_0} \) is nonsingular.

**Proof:** Let \( v \) be a vector in \( T_x \mathbb{R}^n \).

\[
\frac{\partial f}{\partial v}(x) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = (f \circ \gamma)'(0)
\]

where \( \gamma(t) = x + tv \). In particular, the gradient \( Df \) is well defined. We leave showing continuity of this and the higher order derivatives (as well as their existence) as an exercise. The same reasoning does provide, however, the existence of \( Df^{-1} \) as well.

We have, however, that \( df_{x_0}(v) = Df(x_0) \cdot v \) for \( |v| = 1 \). Noting that \( \gamma(t) = f^{-1} \circ f \circ \gamma \) with \( \gamma \) given above in the case \( x = x_0 \), we have

\[
\gamma'(0) = Df^{-1}(f(x_0)) \cdot Df(x_0) \cdot v.
\]

If \( df_{x_0}(v) = Df(x_0) \cdot v = 0 \), then \( v = \gamma'(0) = 0 \), which is a contradiction. Thus, \( df_{x_0} \) is nonsingular. \( \Box \)

In particular, our requirement concerning smooth transfer of curves implies the smoothness of the overlap charts. This observation motivates the general definition of a differentiable manifold.
Definition 2 A differentiable manifold is a topological manifold $M$ with a specified subcollection of charts $\mathcal{A}_0$ whose coordinate neighborhoods still cover $M$ and have differentiable overlaps, i.e.,
\[
\begin{align*}
\{ \xi \circ \xi^{-1} : \xi(U \cap \tilde{U}) &\to \tilde{V} \\
\xi \circ \tilde{\xi}^{-1} : \tilde{\xi}(U \cap \tilde{U}) &\to V
\end{align*}
\]
are smooth maps on subsets of $\mathbb{R}^n$.

$\mathcal{A}_0$ is called an atlas or differentiable structure for $M$. Let $\mathcal{A}$ be the collection of all charts that have smooth overlaps with all the charts in $\mathcal{A}_0$. $\mathcal{A}$ is called the maximal atlas for $M$.

We can give a name, say, $\mathcal{C}$ to the collection of all charts on a topological manifold $M$. We then have $\mathcal{C} \supset \mathcal{A} \supset \mathcal{A}_0$.

Exercise 2 Show that $\mathcal{C}$ can never define a differentiable structure on $M$, that is, the first inclusion above is always strict.

Returning to our observers John Oh and John Wun, this assumption of the existence of smooth overlaps (which follows from the smooth transfer of curves) suggests something interesting about computing distances in $\Sigma$. In particular, we wish to consider how actual distances in $\Sigma$ might be accurately computed in a chart. The path John Wun sees from $p_1$ to $p_2$ corresponds to a straight line on his chart of length $d_2 = d(p_1, p_2)$ which we can parameterize by $\tilde{\gamma}_2(t) = tw_2/|w_2|$ on $[0, d_2]$. This corresponds to some curve $t \mapsto \xi \circ \eta^{-1}(tw_2/|w_2|)$ seen by John Oh in his chart. We would like to define a curve $\gamma$ by this formula and compute the length as John Oh would compute it. Unfortunately due to the stretching that seems to be taking place, even if John Oh were able to pick the correct curve in his chart, it is exceedingly unlikely that he would pick the parameterization given above. For this reason it behoves us to consider something slightly more general. We assume that John Oh has indeed chosen the correct curve (corresponding to the shortest path from $p_1$ to $p_2$ in $\Sigma$) but has parameterized it in some other way $\gamma : [a, b] \to \mathbb{R}^3$. In this situation, the actual length may still be computed as follows.

\[
d_2 = d(p_1, p_2) = \int_a^b \left| \frac{d}{dt}[\eta \circ \xi^{-1} \circ \gamma](t) \right| dt
\]
\[
= \int_a^b \left| d(\eta \circ \xi^{-1})_{\gamma(t)} \cdot \gamma'(t) \right| dt
\]
\[
= \int_a^b \left| d(\eta \circ \xi^{-1})_{\gamma(t)} \cdot \frac{\gamma'(t)}{|\gamma'(t)|} \right| |\gamma'(t)| dt.
\]
Setting

\[ \mu = \left| d(\eta \circ \xi^{-1})_{\gamma(t)} \cdot \frac{\gamma'(t)}{|\gamma'(t)|} \right|, \]

we find that our calculation takes the form

\[ d(p_1, p_2) = \int_a^b \mu(\gamma(t), \gamma'(t))|\gamma'(t)|dt. \] (3)

This is a nice way of expressing the stretching of shortest paths that takes place between the imaginary space \( \mathbb{R}^3 \) of a chart and the physical space \( \Sigma \). More generally, we expect that given any path in a chart, the length of the corresponding path in \( \Sigma \) should be computable by a formula of the form (3). That is, we wish to consider the possibility that given an arbitrary curve \( \gamma \),

\[ \text{length}(\xi^{-1} \circ \gamma) = \int_a^b \mu(\gamma(t), \gamma'(t))|\gamma'(t)|dt. \]

Before we attempt to understand the nature of the function \( \mu \) in general, we introduce some colorful terminology in the abstract setting of differentiable manifolds.

### 2.3 Paths and Curves

We now draw a (somewhat subtle) distinction between the physical path in \( \Sigma \) observed by John Oh or John Won and the imaginary pictures they draw corresponding to it. We will designate what they imagine as curves. Thus, we distinguish between a path in \( \Sigma \) and a curve in \( \mathbb{R}^n \). In the abstract setting of differentiable manifolds, this requires a definition (as it does for curves in \( \mathbb{R}^n \)).

**Definition 3** A curve in \( \mathbb{R}^n \) is a continuous map of an interval \( \gamma : [a, b] \to \mathbb{R}^n \).

A path in an abstract manifold \( M \) is a continuous map of an interval \( \alpha : [a, b] \to M \).

A curve in \( \mathbb{R}^n \) is a smooth map of an interval \( \gamma : [a, b] \to \mathbb{R}^n \).

A smooth path in an abstract manifold \( M \) is a map of an interval \( \alpha : [a, b] \to M \) such that for any coordinate patch \( \xi : \mathcal{U} \to \mathbb{R}^n \) with \( \alpha(t) \in \mathcal{U} \), the map \( \xi \circ \alpha \) is a smooth curve defined in a neighborhood of \( t \).

We denote the collection of all smooth paths in \( M \) by \( \mathcal{P} = \mathcal{P}(M) \) and the subcollection of all smooth paths passing through a particular point \( p \in M \) by \( \mathcal{P}_p \).

Now, we return to physical space \( \Sigma \) and consider some paths and curves. The John’s now add to their imaginary repertoire postulated overlap maps \( \xi \circ \eta^{-1} \) and \( \eta \circ \xi^{-1} \). We will look at the explicit expressions for these maps later, but for now, we let \( \gamma \) be any parameterization of the image of \( t \mapsto \xi \circ \eta^{-1}(tw^2/|w_2|) \) and note an interesting computation:
\[
d(p_1, p_2) = \int_a^b \left| \frac{d}{dt} [\eta \circ \xi^{-1} \circ \gamma](t) \right| dt \\
= \int_a^b \left| d(\eta \circ \xi^{-1})_{\gamma(t)} \gamma'(t) \right| dt \\
= \int_a^b \left| d(\eta \circ \xi^{-1})_{\gamma(t)} \frac{\gamma'(t)}{\gamma'(t)} \right| \gamma'(t) dt \\
= \int_a^b \mu(\gamma(t), \gamma'(t)) \gamma'(t) dt.
\]

2.4 Making a Connection

So far, we have two lines of presentation. There is the physical space \( \Sigma \) which we have called a Riemannian space. This is the space in which John Oh and John Wun live. Our two observers imagine mathematical pictures to visualize and, most importantly, make calculations concerning their physical space. This is a curious situation, because \( \Sigma \) can be physically observed, but not immediately imagined. On the other hand, we have presented the abstract mathematical notions of a topological manifold and a differentiable manifold which isolate certain features of the John’s imaginary pictures. We will now cross the line and assume our mathematical/imaginary pictures can accurately describe the physical space \( \Sigma \). To be sure, we have not yet accomplished this goal. In particular, we should determine the nature of the scaling function \( \mu = \mu(\gamma(t), \gamma'(t)) \) which allows one to compute the lengths of paths. We have now accumulated enough structure, however, that it is convenient to discuss several topics before moving on to consider \( \mu \). A significant portion of what we are doing is introducing colorful terminology that allows our mathematical pictures to invade physical space. We have already seen a good example of this in the definition of paths above. We now move on to the notion of a tangent space in a manifold.

2.5 Smooth Functions and Tangent Space

We first put smooth functions on a manifold in much the same way we did curves.

**Definition 4** A function \( f : M \to \mathbb{R} \) on a differentiable manifold \( M \) is said to be smooth if \( f \circ \xi^{-1} : U \to \mathbb{R} \) is a smooth mapping on \( U \subset \mathbb{R}^n \). The set of all smooth functions on \( M \) is usually denoted by \( \mathcal{D} \) or \( \mathcal{F} \).

Note that \( \mathcal{F} \) is a vector field over \( \mathbb{R} \) and is also a ring.

**Exercise 3** What makes the collection of smooth functions on a manifold a ring? A vector space?
There are various ways to define tangent vectors and tangent spaces on a manifold. They all have fundamentally to do with directional derivatives of functions from $\mathcal{F}$. We will take a somewhat different approach and define tangent vectors at once abstractly and yet as real vectors in $\mathbb{R}^n$.

**Definition 5** Associated to each point $p \in M$ is an abstract $n$-dimensional vector space called the tangent space of $M$ at $p$. The tangent space is denoted by $\mathbb{R}^n_p$ or $T_p M$. A tangent vector is simply an element $V$ of the tangent space. Notice that we have not yet specified coordinates on $\mathbb{R}^n_p = T_p M$. We require $T_p M$ to admit coordinates compatible with the charts of $M$. To be precise, associated to each chart $\xi : U \to \mathbb{R}^n$ with $p \in U$, there is a nonsingular linear map

$$d\xi_p : T_p M \to T_{\xi(p)} \mathbb{R}^n$$

and if $\xi : U \to \mathbb{R}^n$ and $\tilde{\xi} : \tilde{U} \to \mathbb{R}^n$ are charts with $p \in U \cup \tilde{U}$, then

$$d\tilde{\xi}_p \circ d\xi_p^{-1}(v) = \left. \frac{d}{dt} (\tilde{\xi} \circ \xi^{-1})(\xi(p) + tv) \right|_{t=0}$$

for all $v \in T_{\xi(p)} \mathbb{R}^n$.

The table in Figure 4 displays that our definition of tangent vector is not so different from that of a point in a differentiable manifold. The point of view we have taken is that real/physical space exists but does not lend itself to (Euclidean/Cartesian) calculations. Calculations can
<table>
<thead>
<tr>
<th>Objects</th>
<th>Central Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \in M$</td>
<td>\exists \text{ chart such that } p \text{ has an image in } \mathbb{R}^n; charts overlap smoothly</td>
</tr>
<tr>
<td>$\alpha \in \mathcal{P}$</td>
<td>$\xi \circ \alpha\big</td>
</tr>
<tr>
<td>$f \in \mathcal{F}$</td>
<td>$f \circ \xi^{-1}$ is smooth</td>
</tr>
<tr>
<td>$V \in T_p M$</td>
<td>\exists \text{ linear chart such that } V \text{ has an image in } T_{\xi(p)} \mathbb{R}^n; charts overlap with compatibility</td>
</tr>
</tbody>
</table>

$\nabla_V : \mathcal{F} \to \mathbb{R}^n$

\nabla_V f = D_{d\xi_p V} f \circ \xi^{-1}

Figure 4: Objects defined in, on, and around a differentiable manifold.

only be made using imaginary pictures of (possibly small) pieces of physical space. In this sense (the sense of calculation) we cannot see physical space directly. The same limitation presents itself in our notion of tangent vector.

**Definition 6** Given, $V \in T_p M$, the directional derivative of a smooth function $f : M \to \mathbb{R}$ is defined to be

$$\nabla_V f = D_{d\xi_p V} f \circ \xi^{-1}.$$  

**Exercise 4** Show that directional differentiation is well defined by showing that

$$D_{d\xi_p V} f \circ \xi^{-1} = D_{d\xi_p V} f \circ \xi^{-1}.$$  

Solution:

$$D_{d\xi_p V} f \circ \xi^{-1} = \frac{d}{dt} \left[ f \circ \xi^{-1}(\xi(p) + td\xi_p V) \right]_{t=0}$$

$$= D(f \circ \dot{\xi}^{-1})(\dot{\xi}(p)) \cdot \frac{d}{dt} \left[ \xi \circ \xi^{-1}(\xi(p) + td\xi_p V) \right]_{t=0}$$

$$= D(f \circ \dot{\xi}^{-1})(\dot{\xi}(p)) \cdot d\xi_p \cdot d\xi_p^{-1}(d\xi_p V)$$

$$= D(f \circ \dot{\xi}^{-1})(\dot{\xi}(p)) \cdot d\xi_p V$$

$$= D_{d\xi_p V} f \circ \dot{\xi}^{-1}.$$
The third equality uses the compatibility condition defining vectors; the last one uses the definition of directional differentiation. The others involve differentiation in $\mathbb{R}^n$ or are identities. □

**Lemma 3** The directional derivative $\nabla_V : \mathcal{F} \to \mathbb{R}$ is $\mathbb{R}$-linear and Leibnizian, i.e., $\nabla_V(fg) = \nabla_V f g(p) + f(p) \nabla_V g$.

Proof: Note that

$$D_{d\xi_p} f \circ \xi^{-1} = D(f \circ \xi^{-1})(\xi(p)) \cdot d\xi_p V$$

$$= \left( \frac{\partial (f \circ \xi^{-1})}{\partial u^1}(\xi(p)), \ldots, \frac{\partial (f \circ \xi^{-1})}{\partial u^n}(\xi(p)) \right) \cdot d\xi_p V. \quad (4)$$

Thus, since

1. partial differentiation is linear,

2. $\mathbb{R}^n$ is a vector space, and
3. Vector multiplication is distributive, it is clear that $\nabla_V$ is linear. To be precise,

$$
\nabla_V (af + bg) = \ldots \\
= \ldots \\
= a \nabla_V f + b \nabla_V g.
$$

Similarly, since partial differentiation of functions on $\mathbb{R}^n$ is Leibnizian,

$$
\nabla_V (fg) = \ldots \\
= \ldots \\
= \nabla_V f \, g(p) + f(p) \nabla_V g.
$$

**Lemma 4** When considered as a function of $V$ and $f$ on $T_pM \times \mathcal{F}$, the directional derivative $\nabla$ is also linear in $V$.

Proof: This follows from the expression (4) and the linearity of the differential chart $d\xi_p$. □

### 2.5.1 Alternative Formulations

We next consider a catalog of objects isomorphic to $T_pM$. Recall that the collection of all smooth paths passing through $p$ is denoted by $\mathcal{P}_p$. Let us also introduce a subscript convention for curves in $\mathcal{P}_p$, namely, by writing $\alpha_\ast \in \mathcal{P}_p$, we will assume there is a designated value $t_\ast$ for which that $\alpha_\ast(t_\ast) = p$.

**Definition 7 (Equivalence Classes of Paths)** Given a path $\alpha_0 \in \mathcal{P}_p$ with $\alpha_0(t_0) = p$, we define the path vector $\nabla^p$ to be the set

$$
\nabla^p = \{ \alpha_\ast \in \mathcal{P}_p : (\alpha_\ast \circ \xi)'(t_\ast) = (\alpha_0 \circ \xi)'(t_0) \}.
$$

The collection of all distinct path vectors we denote by $\nabla^p$ and call it the path vector space.

**Exercise 5**

1. The definition of $\nabla^p$ does not depend of the chart $\xi : U \to \mathbb{R}^n$ as long as $p \in U$.

2. If $\nabla^p_{\alpha_0} \cap \nabla^p_{\alpha_*} \neq \emptyset$, then $\nabla^p_{\alpha_0} = \nabla^p_{\alpha_*}$. Thus, the sets $\nabla^p$ partition $\mathcal{P}_p$ and $\nabla^p$ denotes precisely the collection of equivalence classes $\nabla^p_{\alpha_0} = [\alpha_0]$
3. The condition

\[ (\alpha_\ast \circ \xi)'(t_\ast) = (\alpha_0 \circ \xi)'(t_0) \tag{5} \]

is an equivalence relation on \( P \), i.e., if we say \( \alpha_\ast \sim \alpha_0 \) when (5) holds, then

(i) \( \alpha_\ast \sim \alpha_\ast \) (always),
(ii) \( \alpha_\ast \sim \alpha_0 \) implies \( \alpha_0 \sim \alpha_\ast \), and
(iii) \( \alpha_\ast \sim \alpha_0 \) and \( \alpha_0 \sim \alpha_1 \) implies \( \alpha_\ast \sim \alpha_1 \).

An equivalence relation always partitions a set into equivalence classes.

We can define addition and multiplication by real scalars on \( P \):

\[ P_{\alpha_1} + P_{\alpha_2} = \{ \alpha_\ast \in P \colon (\alpha_\ast \circ \xi)'(t_\ast) = (\alpha_1 \circ \xi)'(t_1) + (\alpha_2 \circ \xi)'(t_2) \} \]

and

\[ c P_{\alpha_0} = \{ \alpha_\ast \in P \colon (\alpha_\ast \circ \xi)'(t_\ast) = c(\alpha_0 \circ \xi)'(t_0) \} \]

**Exercise 6** Show that these operations are well defined, e.g., exhibit a particular curve \( \alpha_0 \) such that \( P_{\alpha_1} + P_{\alpha_2} = \llbracket \alpha_0 \rrbracket \).

**Exercise 7** \( P \) is an \( n \)-dimensional vector space. What is a basis for \( P \)?

**Exercise 8** \( \phi : T_pM \to P \) by \( V \mapsto \llbracket \alpha_0 \rrbracket \) where

\[ \alpha_0(t) = \xi^{-1}(\xi(p) + td\xi_p V) \]

is a vector space isomorphism with inverse

\[ \phi^{-1} : \llbracket \alpha_0 \rrbracket \mapsto (d\xi_p)^{-1}(d(\alpha_0 \circ \xi)'(t_0)) \]

**Definition 8** (\( \mathbb{R} \)-linear, Leibnizian functionals) Let \( \check{\mathcal{V}} \) be the collection of all \( \mathbb{R} \)-linear, Leibnizian (with respect to \( p \)) functionals \( \check{\mathcal{V}} : \mathcal{F} \to \mathbb{R} \).

**Exercise 9** Show that \( \mathcal{V} \) is a vector space and that \( \phi : T_pM \to \mathcal{V} \) by \( \phi(V) : \mathcal{F} \to \mathbb{R} \) by

\[ \phi(V)(f) = D_{d\xi_p(V)} f \circ \xi^{-1} \]

is a vector space isomorphism with inverse

\[ \phi^{-1} : \mathcal{V} \mapsto (d\xi_p)^{-1} \left( \sum_{j=1}^{n} \check{V} (\xi^j)e_j \right) \].
It is convenient to have an explicit canonical basis (in coordinates) for \( \mathcal{L} \). We begin by defining

\[
\frac{\partial}{\partial \xi^j} \bigg|_p : \mathcal{F} \to \mathbb{R}
\]

by

\[
\frac{\partial}{\partial \xi^j} \bigg|_p (f) = \frac{\partial}{\partial u}[f \circ \xi^{-1}](\xi(p)).
\]

**Exercise 10** Show that

\[
\frac{\partial}{\partial \xi^j} \bigg|_p \in \mathcal{L}.
\]

The tricky part of Exercise 9 is showing that

\[
\left\{ \frac{\partial}{\partial \xi^1} \bigg|_p, \ldots, \frac{\partial}{\partial \xi^n} \bigg|_p \right\}
\]

is a basis for \( \mathcal{L} \). This can be done as follows.

... 

**Definition 9** (do Carmo’s hybrid approach)

\( \mathcal{V} = \left\{ \mathcal{V} : \mathcal{F} \to \mathbb{R} \text{ such that } \mathcal{V}(f) = (f \circ \alpha)'(0) \text{ for some path } \alpha : (-\epsilon, \epsilon) \to M \right\} \).

**Exercise 11** Prove that \( \mathcal{L} = \mathcal{V} \).

In light of this exercise, we can henceforth ignore the distinction between \( \mathcal{V} \) and \( \mathcal{L} \) and use both the corresponding paths and linear Leibnizian properties of such functionals.
Episode II: What They Figured

We return now to our observers John Oh and John Wun. After arguing about the shortest path from $p_1$ to $p_2$, they decide to check a sequence of twenty points along the straight line in John Oh’s chart from $u_1$ to $u_2$. These points are $q_j = \xi^{-1}(d_1, jd_1/20)$ corresponding to $u_j = (d_1, jd_1/20)$ for $j = 1, \ldots, 20$. See Figure 6.

They found that each point $q_j$ is slightly closer to John Wun (at $p_1$) than the distance $t_j = jd_1/20$ along John Oh’s line. Here is a table with each measurement and the ratio of the two.

<table>
<thead>
<tr>
<th>$t_j = jd_1/20$</th>
<th>$d(p_1, q_j)$</th>
<th>$t_j/d(p_1, q_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>39269.908</td>
<td>35355.020</td>
<td>.900308</td>
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<td>78539.816</td>
<td>70708.128</td>
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</tbody>
</table>

Table 1: Ratios of lengths of curves measured in charts
With these measurements in mind, recall our computation of the lengths of paths in a
chart (3) and let \( \gamma(t) = u_1 + te_2 \). Putting the two together, we have

\[
\text{length}(\xi^{-1} \circ \gamma) = \int_0^{d_1} \mu(\gamma(t), \gamma'(t)) dt.
\]

Note that

\[
\mu(\gamma(0), \gamma'(0)) = \mu(u_1, e_2) = |D(\eta \circ \xi^{-1})(u_1) \cdot e_2|
\]

\[
= \left| \frac{d}{dt} [\eta \circ \xi^{-1}(u_1 + te_2)] \right|_{t=0}
\]

\[
= \lim_{t \to 0} \frac{\eta \circ \xi^{-1}(u_1 + te_2) - 0}{t}
\]

\[
= \lim_{t \to 0} \frac{\eta \circ \xi^{-1}(u_1 + te_2)}{t}.
\]

More generally,

\[
\mu(u_1, ae_1 + be_2) = |D(\eta \circ \xi^{-1})(u_1) \cdot (ae_1 + be_2)|
\]

\[
= |aD(\eta \circ \xi^{-1})(u_1) \cdot e_1 + bD(\eta \circ \xi^{-1})(u_1) \cdot e_2|
\]

\[
= (a^2 Le_1 \cdot Le_1 + 2ab Le_1 \cdot Le_2 + b^2 Le_2 \cdot Le_2)^{1/2}
\]

where \( Lv = D(\eta \circ \xi^{-1})(u_1) \cdot v \).

**Exercise 12** Compute \( Le_1 \) and conjecture from Figure 6 and the measured values in Table 1 the value of \( Le_2 \).

**Exercise 13** What measurements should be undertaken to determine \( \mu(u, v) \) in general?

### 2.6 Vector Fields; Tangent Manifold and Smooth Mappings of Manifolds

In our discussion of tangent space at \( p \in M \), we should have indexed all things by \( p \): \( \mathcal{P}, \mathcal{V}_p, \mathcal{P}, \mathcal{P}, \mathcal{V}_p, \mathcal{V}_p \). One should take particular note of the basis

\[
d\xi^{-1}_p(e_1), \ldots, d\xi^{-1}_p(e_n)
\]

corresponding to bases

\[
[t \mapsto \xi^{-1}(\xi(p) + te_1)], \ldots, [t \mapsto \xi^{-1}(\xi(p) + te_n)]
\]
and
\[
\frac{\partial}{\partial \xi^1}_p, \ldots, \frac{\partial}{\partial \xi^n}_p
\]
for \( \mathcal{V} \) and \( \mathcal{\hat{V}} \) respectively. In fact, these coordinate basis expressions are valid at other points in \( U \):
\[
d\xi_q^{-1}(e_1), \ldots, d\xi_q^{-1}(e_n)
\]
is a basis for \( T_qM \) whenever \( q \in U \). Each such coordinate vector field, or any sum of them
\[
V(p) = \sum a_j(p)d\xi_p^{-1}(e_j)
\]
corresponds to a vector field
\[
\mathcal{U} \rightarrow \sum a_j \circ \xi^{-1}(u)e_j = (a_1 \circ \xi^{-1}(u), \ldots, a_n \circ \xi^{-1}(u))
\]
on \( \xi(U) \). Technically, \( V \) is not really a vector field on \( M \), but the difficulty is similar to that with referring to \( \xi^j \) as a function in \( \mathcal{F}(M) \). What we really have is a vector field on \( U \) that can be taken as the germ of a vector field at any point in \( U \) (see Exercise 15 below).

**Definition 10** The tangent manifold (or tangent bundle) is
\[
TM = \cup_{p \in M} T_pM.
\]
A vector field is a function \( V : M \rightarrow TM \) such that
\[
V(p) \in T_pM.
\]
Naturally, \( V(p) \) is often denoted \( V_p \).

Given a vector field, one obtains an expression of the form (6) for each coordinate chart \( \xi : U \rightarrow \mathbb{R}^n \). The vector field \( V \) is said to be a smooth vector field if the functions \( a_j : U \rightarrow \mathbb{R} \) are smooth, i.e., are elements of \( \mathcal{F}(U) \), for every \( \xi \). We will denote the collection of smooth vector fields by \( \mathcal{V} = \mathcal{V}(M) \).

**Exercise 14** Let \( V \) be a vector field on \( M \).

1. Show that given a coordinate chart \( \xi : U \rightarrow \mathbb{R}^n \), the vector field has a unique expression (6) for \( p \in U \) with
\[
a_j(p) = d\xi_p(V(p)) \cdot e_j.
\]
2. Show that $V$ is smooth if and only if

$$v(u) = d_{\xi^{-1}(u)}(V(\xi^{-1}(u)))$$

defines a smooth vector field on $U \subset \mathbb{R}^n$ (with the usual definition that the coordinate functions are smooth). Key identity:

$$v^j(u) = d_{\xi^{-1}(u)}(V(\xi^{-1}(u))) \cdot e_j$$

$$= d_{\xi^{-1}(u)} \left( \sum a_k(\xi^{-1}(u))d_{\xi^{-1}(u)}(e_k) \right) \cdot e_j$$

$$= \sum a_k(\xi^{-1}(u))d_{\xi^{-1}(u)} \circ d_{\xi^{-1}(u)}(e_k) \cdot e_j$$

$$= \sum a_k \circ \xi^{-1}(u)e_k \cdot e_j$$

$$= a_j \circ \xi^{-1}(u).$$

3. Show that $V$ is smooth if and only if the function $g(p) = \bar{V}_p(f)$ is a smooth function for each $f \in \mathcal{F}(M)$ where $\bar{V}_p$ is the unique linear Leibnizian functional corresponding to $V(p)$, i.e., show that $\bar{V}: \mathcal{F}(M) \to \mathcal{F}(M)$.

4. A ring operator $\bar{V}: \mathcal{F}(M) \to \mathcal{F}(M)$ is said to be $\mathbb{R}$-linear and Leibnizian if

(i) $\bar{V}(af + bg) = a \bar{V}(f) + b \bar{V}(g)$ for every $f, g \in \mathcal{F}$ and $a, b \in \mathbb{R}$ and

(ii) $\bar{V}(fg) = \bar{V}(f)g + f \bar{V}(g)$ for all $f, g \in \mathcal{F}$.

Show that every $\mathbb{R}$-linear and Leibnizian operator corresponds to directional differentiation with respect to some vector field. Note: An $\mathbb{R}$-linear and Leibnizian operator on $\mathcal{F}$ is called a (functional) derivation on $M$.

Exercise 15 Given $V \in \mathcal{V}(U)$ and a coordinate chart $\xi: U \to \mathbb{R}^n$, show that for each $p \in U$, there is a vector field $\bar{V} \in \mathcal{V}$ with $V \equiv \bar{V}$ in some neighborhood of $p$.

Exercise 16 Explain what it means for $\mathcal{V}(M)$ to be a module over the ring $\mathcal{F}(M)$.