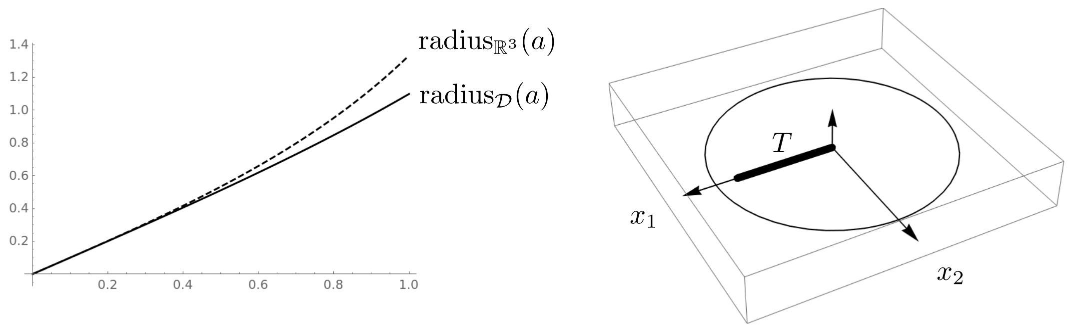

Figure C.2: Comparison of radius$_{\mathcal{D}}(a)$ and radius$_{\mathbb{R}^3}(a)$ (left). Here it is impossible for a tether to reach a circle in a horizontal plane of the appropriate radius radius$_{\mathbb{R}^3}(a)$. The heavy tether segment T of length radius$_{\mathcal{D}}(a)$ extends from the origin and is straining to reach the circle but is too short.

In this case, however, one cannot easily determine a family of tether curves with constant length radius$_{\mathcal{D}}(a)$ connecting the origin to such a curve Γ. If one is to obtain identification with an embedded surface in $\mathbb{R}^3$ even locally, something more complicated must be done. Something is essentially different about these Riemann surfaces.

Exercise C.2. Show the saddle shaped surface

$$\left\{(x_1, x_2, x_1^2 - x_2^2) : (x_1, x_2) \in \mathbb{R}^2\right\}$$

can be used to locally induce a matrix assignment on an open disk $B_\epsilon(\mathbf{0})$ giving a one-to-one correspondence of lengths of paths on the surface with the Riemannian lengths of the corresponding paths in the disk calculated using the induced matrix assignment. Show, however, that this matrix assignment is not axially symmetric in the disk.

C.5 Ruijia's question

In some cases asking a good question can be better than giving an answer. Often a question is not entirely well-posed. Such an ill-posed question can often play a key role in gaining understanding through the process of considering very carefully what the question is really about and asking auxiliary questions concerning what is required to obtain a well-posed question.

Ruijia asked a question which (I believe) is not entirely well-posed but offers some opportunity for learning through the process I've attempted to describe above.

Ruijia's original question was something like this:

> How is it possible to put a *smooth structure* on the surface of a cube?

I think this is a great question, if not entirely well-posed. Let me start first with an auxiliary question which is perhaps not of direct interest, but which I think is a reasonable one. How would you specifiy the surface of a cube precisely in Euclidean coordinates? There is a choice of coordinates here, and I think I'm safe in assuming "ambient" coordinates of dimension three. That is to say, the surface of a cube in mind here is a subset of $\mathbb{R}^3$. One example would be

$$\partial C_1(\mathbf{0}) = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \max\{|x_1|, |x_2|, |x_3|\} = 1\}. \tag{C.2}$$

The quantity

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, |x_3|\}$$

is sometimes called the ℓ^∞ **norm** on $\mathbb{R}^3$. The general notion of a norm (and a normed space) to go along with it are going to be important concepts for us, and they should be covered in the chapter on spaces where you can now read about topological spaces. Because the quantity appearing in the definition of $\partial C_1(\mathbf{0})$ above is a norm (the ℓ^∞ norm on $\mathbb{R}^3$) the set $\partial C_1(\mathbf{0})$ is sometimes called the surface of the ℓ^∞ **unit cube** or the ℓ^∞ unit sphere.[4] Notice one obtains the definition of

$$\mathbb{S}^2 = \partial B_1(\mathbf{0}) = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$$

if the ℓ^∞ norm is repleced in (C.2) with the Euclidean norm

$$|\mathbf{x}| = \sqrt{\sum_{j=1}^{3} x_j^2}$$

which can also be denoted by $|\ \cdot\ | = \|\ \cdot\ \|_2$ and is the ℓ^2 norm on $\mathbb{R}^3$. You may recall that we have called $\mathbb{S}^2$ the **unit sphere**, but what one means by

[4]The symbols $\|\mathbf{x}\|_\infty$ should be read "the ell-infinity norm of $\mathbf{x}$."

that is the unit sphere with respect to the Euclidean (or ℓ^2) norm. Perhaps a little terminology is in order if we want to discuss the surface of a cube and the unit cube in particular. The subset

$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_1 = -1\}$$

is a **face** or closed face of the unit cube. This face is illustrated in Figure C.3. The surface of the cube has five more faces

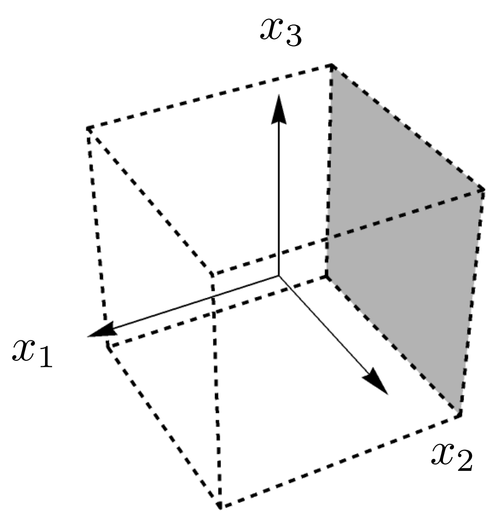

Figure C.3: The unit cube in $\mathbb{R}^3$ and its back face.

$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_1 = 1\}$$
$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_2 = -1\}$$
$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_2 = 1\}$$
$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_3 = -1\}$$
$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_3 = 1\}$$

which may be referred to as the front, left, right, bottom, and top faces respectively. We can also consider the **open faces**

$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_1 = -1,\ \|(x_2, x_3)\|_\infty < 1\}$$
$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_1 = 1,\ \|(x_2, x_3)\|_\infty < 1\}$$
$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_2 = -1,\ \|(x_1, x_3)\|_\infty < 1\}$$
$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_2 = 1,\ \|(x_1, x_3)\|_\infty < 1\}$$
$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_3 = -1,\ \|(x_1, x_2)\|_\infty < 1\}$$
$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_3 = 1,\ \|(x_1, x_2)\|_\infty < 1\}$$

which have nothing to do with sandwiches really.[5]

The complement of an open face with respect to the corresponding closed face consists of four **edges**. For example, the edges of the back face are

$$\begin{aligned}&\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_1 = x_2 = -1\}\\ &\quad\bigcup\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_1 = x_3 = -1\}\\ &\qquad\bigcup\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_1 = -x_2 = -1\}\\ &\qquad\quad\bigcup\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_1 = -x_3 = -1\}.\end{aligned}$$

Alternatively, those closed faces with a nonempty intersection may be called **adjacent faces** and each pair of adjacent faces intersects is a (closed) edge. Thus, the back-right edge is

$$\{\mathbf{x} = (x_1, x_2, x_3) \in \partial C_1(\mathbf{0}) : x_1 = -x_2 = -1\}.$$

opposite faces have empty intersection, and there are three pairs of those: front-and-back, left-and-right, and bottom-and-top.

The intersection terminology as well as the dimensionally relative open-closed terminology can be extended to the edges.

Exercise C.3. How would you specify (write explicitly as a set) the open back-bottom edge?

Exercise C.4. How would you express the condition that the open back-bottom edge is actually an open set topologically? Hint: First consider how an open face can properly be considered (topologically) as an open set.

[5]On a more serious note, you can see that I've used the ℓ^∞ norm on $\mathbb{R}^2$ to specify these open faces.

Finally, adjacent (closed) edges intersect in **vertices**, and there are of course eight of those.

Exercise C.5. What are the relations between edges and topological boundaries (as we have defined edges for a cube above)?

Now that we have one cube and the burden of a significant amount of terminology to go along with it, I'd like to consider other cubes. One way to obtain other cubes (and the surfaces determined by them) is by translation. For example, a unit cube in the first octant is given by

$$\{\mathbf{x} + (1,1,1) : \mathbf{x} \in C_1(\mathbf{0})\}.$$

This one might naturally be referred to as a cube of side (or edge) length two since for example $\{(t,0,0) : 0 \le t \le 2\}$ is in the surface of this cube. Another possibility is scaling. Two cubes of side length one are given by

$$\{\mathbf{x}/2 : \mathbf{x} \in C_1(\mathbf{0})\} \qquad \text{and} \qquad \{\mathbf{x}/2 : \mathbf{x} \in C_1(\mathbf{p})\}$$

where $C_1(\mathbf{p}) = \{\mathbf{x} + \mathbf{p} : \mathbf{x} \in C_1(\mathbf{0})\}$. Finally, there is the possibility of rotation. The group of rotations of $\mathbb{R}^3$ is in one-to-one correspondence with the collection $SL_3(\mathbb{R})$ of 3×3 matrices with real entries and determinant one. In particular, there is a two-parameter family of these rotations—they make a two-dimensional Lie group if you like—and they can be nicely parameterized on the two dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$. I won't get into the details of this at the moment except to say that sometimes the symbol $\mathbb{T}^2$ and the terminology "the two (dimensional) torus" are used to refer to the surface

$$\{(2+\cos t)(\cos s, \sin s, 0) + \sin t(0,0,1) : (s,t) \in [0,2\pi] \times [0,2\pi]\} \subset \mathbb{R}^3.$$

rather than to the surface

$$\begin{aligned} \mathbb{T}^2 &= \mathbb{S}^2 \times \mathbb{S}^2 \\ &= \{\mathbf{x} = (x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : |(x_2,x_2)| = |(x_3,x_4)| = 1\} \\ &\subset \mathbb{R}^4. \end{aligned} \tag{C.3}$$

Exercise C.6. Find bijections between $\mathbb{T}^2$ given in (C.3) and the sets

$$\mathcal{S} = \{(2+\cos t)(\cos s, \sin s, 0) + \sin t(0,0,1) : (s,t) \in \mathbb{R}^2\} \subset \mathbb{R}^3$$

and

$$Z = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}.$$

All three of these sets are referred to as the two-dimensional torus and denoted by $\mathbb{T}^2$ in [2].

Given a specific rotation $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ we can consider the cube

$$\{\psi(\mathbf{x}) : \mathbf{x} \in C_1(\mathbf{0})\}$$

where $C_1(\mathbf{0}) = \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_\infty < 1\}$ and its boundary surface. Combining scaling, rotation, and translation, we can express any cube in $\mathbb{R}^3$ as

$$\{\psi(\alpha\mathbf{x}) + \mathbf{p} : \mathbf{x} \in C_1(\mathbf{0})\}$$

where ψ is a rotation, $\alpha > 0$, and $\mathbf{p} \in \mathbb{R}^3$, and then we can discuss the boundary surface, faces, edges, and vertices of such a cube.

I'd like to turn my attention now to **parameterization** of the surface of a cube. This again may be viewed as something of a tangent to the main content of Ruijia's question, but if I'm correct, his question is not quite well-posed so this sort of meandering around is to be expected.[6]

The portions of $\partial C_1(\mathbf{0})$ within an open face can be easily parameterized on an open subset of $\mathbb{R}^2$. For example, $\mathbf{p} : B_1(\mathbf{0}) \to \partial C_1(\mathbf{0})$ by

$$\mathbf{p}(\mathbf{x}) = (x_1, x_2, -1)$$

parameterizes a portion of the bottom face of $\partial C_1(\mathbf{0})$. See Figure C.4 (left). If you think about it, it should become intuitively clear that the entire surface of the cube cannot be parameterized by a single chart function (homeomorphism) defined on a open chart U in $\mathbb{R}^2$. One explanation for why this is true is the following: If we assume $\mathbf{p} : U \to \partial C_1(\mathbf{0})$ is a global chart function defined on an open set $U \subset \mathbb{R}^2$, then the open set $U \backslash \{\mathbf{x}\}$ obtained by removing a single point from U is not simply connected. On the other hand, the supposed image $\mathbf{p}(B_1(\mathbf{0}) \backslash \{\mathbf{x}\}) = \partial C_1(\mathbf{0}) \backslash \{\mathbf{p}(\mathbf{x})\}$ is simply connected. Since the property of being simply connected is preserved under homeomorphism, this is a contradiction.

Being forced to have more than one chart to parameterize the surface of a cube (or any surface) is a kind of technical complication in this context. In particular, I don't think we need to consider the complication of having more than one chart to address Ruijia's question about the smoothness (or a smooth structure) on the surface of a cube. With this in mind I'm going to now focus on the particular surface

$$\{\mathbf{x}/2 + (1/2, 1/2, 1/2) : \mathbf{x} \in \partial C_1(\mathbf{0})\}$$

[6] If you want a precise answer, then ask a precise question.

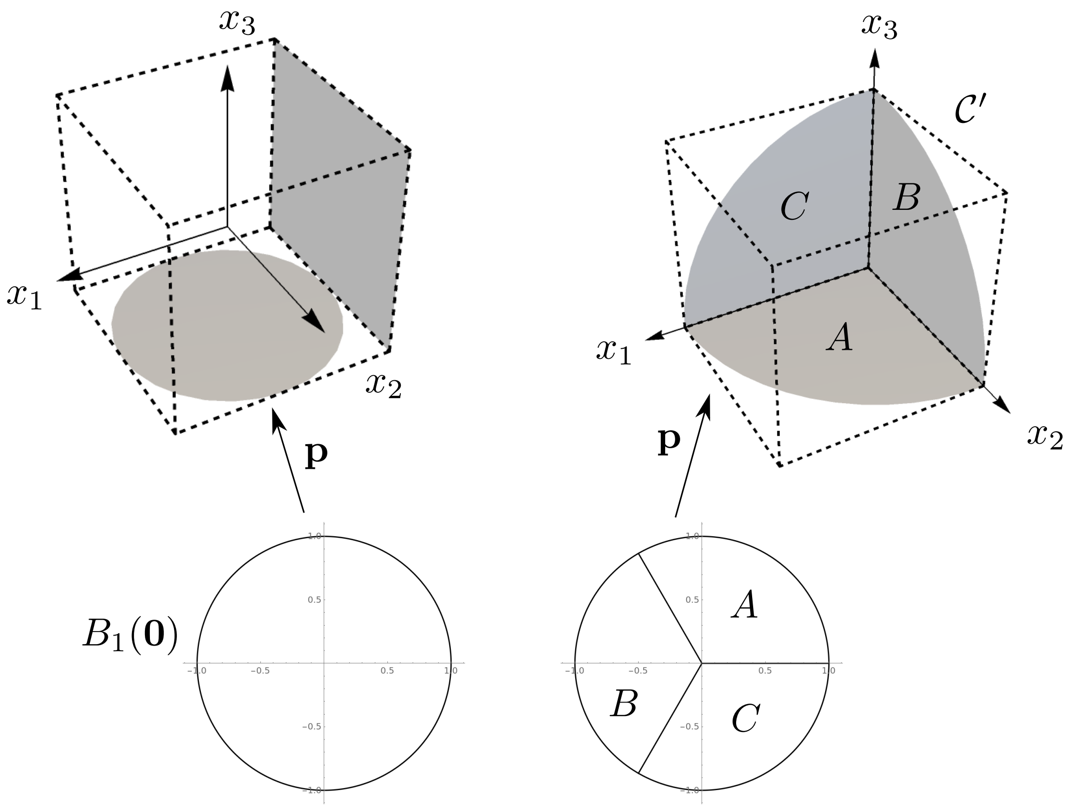

Figure C.4: Continuous chart functions from a unit disk into the surface of a cube. The trihedral corner surface (right)

and specifically the portion of it illustrated on the right in Figure C.4. This surface is the image of the chart function $\mathbf{p} : B_1(\mathbf{0}) \to \mathbb{R}^3$ given by

$$\mathbf{p}(\mathbf{x}) = \begin{cases} (0,0,0), & \mathbf{x} = \mathbf{0} \in \mathbb{R}^2 \\ |\mathbf{x}| \left(\cos\left(\frac{3}{4}\cos^{-1}\frac{x_1}{|\mathbf{x}|}\right), \sin\left(\frac{3}{4}\cos^{-1}\frac{x_1}{|\mathbf{x}|}, 0\right), 0\right) & \mathbf{x} \in A, \\ |\mathbf{x}| \left(0, \cos\left(\frac{3}{4}\sin^{-1}\frac{x_1}{|\mathbf{x}|}\right), \sin\left(\frac{3}{4}\sin^{-1}\frac{x_1}{|\mathbf{x}|}\right)\right), & \mathbf{x} \in B, \\ |\mathbf{x}| \left(\sin\frac{3}{4}\left(\pi - \cos^{-1}\frac{x_1}{|\mathbf{x}|}\right), 0, \cos\frac{3}{4}\left(\pi - \cos^{-1}\frac{x_1}{|\mathbf{x}|}\right)\right), & \mathbf{x} \in C \end{cases}$$

where

$$A = \left\{ \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) : 0 \le x_2,\ 0 < x_1 + x_2,\ -\frac{|\mathbf{x}|}{2} \le x_1 \right\}$$
$$B = \left\{ \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) : x_1 \le -\frac{|\mathbf{x}|}{2} \right\}$$
$$C = \left\{ \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) : x_2 <,\ -\frac{|\mathbf{x}|}{2} \le x_1 \right\}.$$

Figure C.5 may be helpful in understanding the (inverse) trigonometric functions used to write down the formulas for the parameterization/chart function $\mathbf{p}$. Let us call this surface $\mathcal{C}'$.

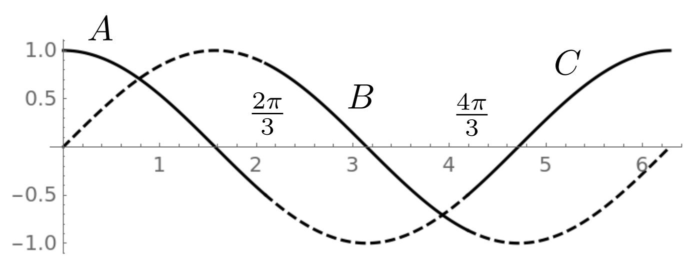

Figure C.5: Plots of cosine and sine and their various pieces used in the construction of the piecewise definition of the chart function $\mathbf{p}$ of the trihedral corner $\mathcal{C}'$.

I suspect Ruijia is worried about the singularities in the surface of the cube along the edges and also at the vertices. It will be noted that the surface $\mathcal{C}'$ also has the same kind of singularities along the edges of the cube and at the trihedral corner. What is true is that it is not possible to (and note this terminology carefully) parameterize $\mathcal{C}'$ with a **regular** C^1 **parameterization** or of course a regular smooth parameterization. Indeed, the parameterization $\mathbf{p}$ is not even C^1. We can only say $\mathbf{p} \in C^0(B_1(\mathbf{0}) \to \mathbb{R}^3)$. It is possible to obtain a C^1 parameterization of $\mathcal{C}'$ or even a C^∞ parameterization of $\mathcal{C}'$, that is a bijective function $\mathbf{q} \in C^\infty(B_1(\mathbf{0}) \to \mathbb{R}^3)$ with $\mathbf{q}(B_1(\mathbf{0})) = \mathcal{C}'$. I have not written one down, but you can ponder how that might be done.

Exercise C.7. Find a smooth parameterization $\mathbf{q}$ of $\mathcal{C}'$ as described above. Hint: Make sure all the derivatives of $\mathbf{q}$ vanish at $\mathbf{0} \in B_1(\mathbf{0})$ and all the angular derivatives of $\mathbf{q}$ vanish along the boundaries of the regions A, B, and C illustrated on the bottom right in Figure C.4.

A **regular** parameterization, however, is something different, and that rules out corners. Here is a definition which is a little technical, but I'll highlight the key part relevant to the discussion above.

Definition 23. (embedded regular surface in $\mathbb{R}^3$) Let $\mathbf{q} \in C^1(V \to \mathcal{S})$ be a diffeomorphism which is a global parameterization of a surface $\mathcal{S} \subset \mathbb{R}^3$ defined on an open set $V \subset \mathbb{R}^2$. By this we mean $\mathbf{q} : V \to \mathcal{S} \subset \mathbb{R}^3$ is a bijective function with coordinate functions $\mathbf{q} = (q^1, q^2, q^3)$ for which all the partial derivatives

$$\frac{\partial q^i}{\partial x_j} \qquad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2$$

satisfy

$$\frac{\partial q^i}{\partial x_j} \in C^0(V)$$

and for which given each point $P \in \mathcal{S}$ there are open sets $U \subset V$ and $W, Q \subset \mathbb{R}^3$ with $P \in W$ and a homeomorphism $\psi : W \to Q$ satisfying the following:

(i) The coordinate functions $\psi = (\psi^1, \psi^2, \psi^3)$ satisfy

$$\frac{\partial \psi^i}{\partial x_j} \in C^0(W). \qquad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3,$$

(ii) The coordinate functions $\phi = \psi^{-1} = (\phi^1, \phi^2, \phi^3)$ of the inverse $\phi : Q \to W$ satisfy

$$\frac{\partial \phi^i}{\partial x_j} \in C^0(Q). \qquad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3,$$

(iii) $U_0 = \{(\mathbf{x}, 0) \in \mathbb{R}^3 : \mathbf{x} \in U\} \subset Q \subset \{(\mathbf{x}, z) \in \mathbb{R}^3 : \mathbf{x} \in U\}$,

(iv) $\phi(U_0) \equiv W \cap \mathcal{S}$, and

(v) The restriction

$$\phi\big|_{U_0} : U_0 \to \mathcal{S} \cap W \qquad \text{satisfies} \qquad \phi\big|_{U_0}(\mathbf{x}, 0) \equiv \mathbf{q}(\mathbf{x}) \quad \text{for } \mathbf{x} \in U_0.$$

With all of the above given, we say $\mathbf{q}$ is **regular** if the vectors

$$\frac{\partial \mathbf{q}}{\partial x_1}(\mathbf{x}) = \left(\frac{\partial q^1}{\partial x_1}(\mathbf{x}), \frac{\partial q^2}{\partial x_1}(\mathbf{x}), \frac{\partial q^3}{\partial x_1}(\mathbf{x}) \right)$$

and

$$\frac{\partial \mathbf{q}}{\partial x_2}(\mathbf{x}) = \left(\frac{\partial q^1}{\partial x_2}(\mathbf{x}), \frac{\partial q^2}{\partial x_2}(\mathbf{x}), \frac{\partial q^3}{\partial x_2}(\mathbf{x}) \right)$$

are linearly independent in $\mathbb{R}^3$ for each $\mathbf{x} \in V$. This is the crucial condition making a parameterization regular, and this is what you cannot get for the trihedral corner.

Most of the conditions described in Definition 23 are illustrated in Figure C.6.

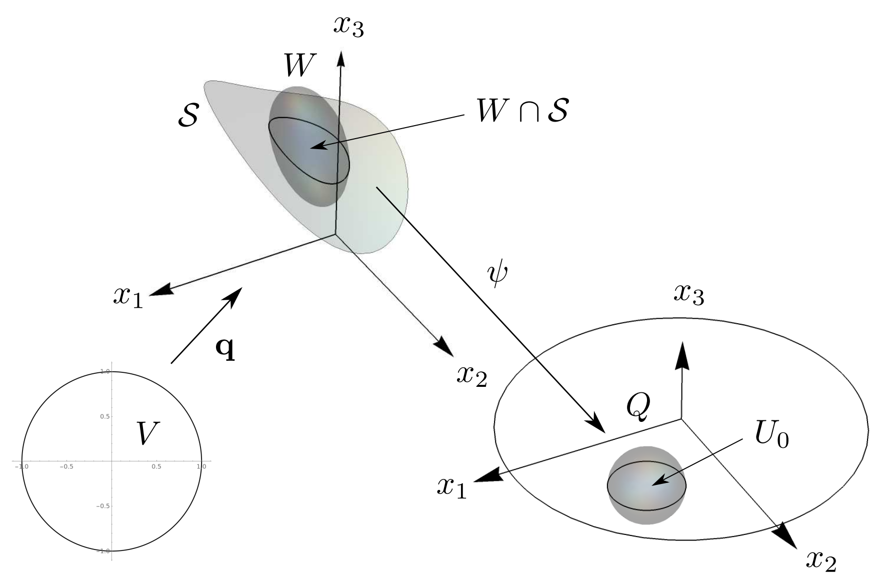

Figure C.6: A nice chart function $\mathbf{q}$ for a regular embedded surface $\mathcal{S} \subset \mathbb{R}^3$.

Note: The conditions on the chart function $\mathbf{q}$, and the complicated conditions **(i)-(v)** in particular in Definition 23 can be simplified substantially, but these more complicated conditions still follow for some restriction of a chart function as long as the crucial "regularity" condition concerning linear independence is retained. I have stated the definition including a chart function with these additional properties because they illustrate in more detail some of the properties of a regular embedded surface and I think they are worth keeping in mind. Specifically, it may simply be assumed that $\mathbf{q}$ is a differentiable homeomorphism for which the two partial derivatives $\mathbf{q}_{x_1}$ and $\mathbf{q}_{x_2}$ are linearly independent at each point.

The definition above can also be generalized somewhat if, rather than assuming a global chart function $\mathbf{q}$, one assumes the existence of a local chart function $\mathbf{q}$ associated with each point $P \in \mathcal{S}$. For comparison on all these points see Definition 1 in Chapter 2 (page 52) and the following section of [1].

Exercise C.8. Explain why the trihedral corner surface $\mathcal{C}'$ is not a regular surface according to Definition 23. Hint: If $\mathbf{q}_{x_1}$ and $\mathbf{q}_{x_2}$ are linearly indepdendent, there should be a well-defined tangent plane at each point $P \in \mathcal{S}$.

I've been calling the trihedral corner surface a "surface," and I've got a definition of an embedded regular surface, but the trihedral corner is not one of those. Perhaps I should offer a definition of some kind of surface that actually applies to the trihedral corner surface $\mathcal{C}'$. The usual approach is to designate a set like $\mathcal{C}'$ a **piecewise affine surface**. Such surfaces might also be referred to as "piecewise linear" or "PL" surfaces. The definition can be a little delicate. In particular, it is apparently somewhat difficult to formulate a reasonably flexible definition in keeping with our convenient restriction of having single global chart function.

The following definition is a work in progress

> I will remark the following: A piecewise affine surface embedded in $\mathbb{R}^3$ should be a more general object than a polyhedral surface or the surface of a polyhedron. Concerning these latter there is a nice quote of Branko Grünbaum [3]:
>
> > The Original Sin in the theory of polyhedra goes back to Euclid, and through Kepler, Poinsot, Cauchy and many others continues to afflict all the work on this topic (including that of the present author). It arises from the fact that the traditional usage of the term "regular polyhedra" was, and is, contrary to syntax

> and to logic: the words seem to imply that we are dealing, among the objects we call "polyhedra", with those special ones that deserve to be called "regular". But at each stage—Euclid, Kepler, Poinsot, Hess, Briickner,. . . —the writers failed to define what are the "polyhedra" among which they are finding the "regular" ones. True, we now know what are the convex polyhedra, which we think are the polyhedra Euclid had in mind; hence there is no stigma attached to the use of a term like "regular convex polyhedron". But where in the literature do we find acceptable definitions of polyhedra that could be specialized to give the "regular Kepler-Poinsot polyhedra"? For these, a better expression would be to say that they are "regularpolyhedra"—a distinct kind of objects, constructed according to more or less explicit procedures, and without any connection to what the separate parts of that ungainly word may mean.

Definition 24. (piecewise affine surface embedded in $\mathbb{R}^3$) Let $V_1, V_2, \ldots, V_k$ be finitely many disjoint simply connected open sets in $\mathbb{R}^2$ with the following properties:

F1 $\overline{V_i} \cap \overline{V_j} = \phi$ for $i \neq j$,

F2 Each V_i for $i = 1, 2, \ldots, k$ has boundary

$$\partial V_i = \bigcup_{j=1}^{\ell_i} \Gamma_{ij}$$

with Γ_{ij} for $j = 1, 2, \ldots, \ell_i$ a regularly parameterized curve parameterized with a parameterization $\gamma_{ij} \in C^1([a_{ij}, b_{ij}] \to \Gamma_{ij})$ with

F3 $|\gamma'_{ij}| = 1$, (arclength parameterizations)

F4 $\gamma_{ij}(b_{ij}) = \gamma_{i,j+1}(a_{i,j+1})$ for $j = 1, 2, \ldots, \ell_i - 1$, (concatenation)

F5 The concatenation $\Gamma_{i1}, \Gamma_{i2}, \ldots, \Gamma_{i\ell_i}$ is a simple closed curve with

$$\gamma_{i\ell_i}(b_{i\ell_i}) = \gamma_{i1}(a_{i1}),$$

(concatenation is closed) and

F6 For at least one $\mathbf{x} \in \partial V_i$ with $\gamma_{ij}(t) = \mathbf{x}$ there holds

$$\mathbf{x} - s[\gamma'_{ij}(t)]^{\perp} = \mathbf{x} - s\left(-(\gamma'_{ij})^2(t), (\gamma'_{ij})^1(t)\right) \notin V_i, \qquad \text{for } s \geq 0$$

where $\gamma_{ij} = (\gamma^1_{ij}, \gamma^2_{ij})$ and

$$\gamma'_{ij} = \left((\gamma^1_{ij})', (\gamma^2_{ij})'\right) = \left((\gamma'_{ij})^1, (\gamma'_{ij})^2\right).$$

That is, ∂V_i is parameterized in the counterclockwise direction.

Let $\mathcal{S} \subset \mathbb{R}^3$ and assume there are chart functions $\mathbf{q}^i \in C^\infty(V_i \to \mathcal{S})$ for $i = 1, 2, \ldots, \ell_i$ satisfying the following:

C1 $\mathbf{q}^i$ is affine in the sense that

$$\mathbf{q}^i(\mathbf{x}) = (A_i \mathbf{x}^T)^T + \mathbf{b}_i \tag{C.4}$$

where A_i is a 3×2 matrix with rank two, i.e., the constant vectors

$$\frac{\partial \mathbf{q}^i}{\partial x_1}(\mathbf{x}) = \left(\frac{\partial q^{i1}}{\partial x_1}(\mathbf{x}), \frac{\partial q^{i2}}{\partial x_1}(\mathbf{x}), \frac{\partial q^{i3}}{\partial x_1}(\mathbf{x})\right)$$

and

$$\frac{\partial \mathbf{q}^i}{\partial x_2}(\mathbf{x}) = \left(\frac{\partial q^{i1}}{\partial x_2}(\mathbf{x}), \frac{\partial q^{i2}}{\partial x_2}(\mathbf{x}), \frac{\partial q^{i3}}{\partial x_2}(\mathbf{x})\right)$$

are linearly independent in $\mathbb{R}^3$, and $\mathbf{b}_i \in \mathbb{R}^3$ is a fixed affine shift vector. (The use of the double transpose in (C.4) allows the mixing of row vectors and standard matrix multiplication.)

C2 Note each chart function $\mathbf{q}^i$ for $i = 1, 2, \ldots, k$ extends by the formula (C.4) not only to the closure $\overline{V_i}$ but to all of $\mathbb{R}^2$. We let $\mathbf{q}^i \in C^\infty(\mathbb{R}^2 \to \mathbb{R}^3)$ denote the extension as well. On the other hand, $\mathbf{q}^i : \mathbb{R}^2 \to \mathbf{q}^i(\mathbb{R}^2)$ is a homeomorphism, and we denote by $(\mathbf{q}^i)^{-1}$ the inverse of the restriction to $\overline{V_i}$ so that

$$(\mathbf{q}^i)^{-1} = \left(\mathbf{q}^i\big|_{\overline{V_i}}\right)^{-1} : \mathbf{q}^i(\overline{V_i}) \to \overline{V_i}.$$

For $i = 1, 2, \ldots, k$, let

$$\mathcal{I}_i = \{m \in \{1, 2, \ldots, k\} \backslash \{i\} : \mathbf{q}^i(\overline{V_i}) \cap \mathbf{q}^m(\overline{V_m}) \neq \phi\}.$$

For each $m \in \mathcal{I}_i$, we require

$$\mathbf{q}^i(\overline{V_i}) \cap \mathbf{q}^m(\overline{V_m}) = \Gamma_{ij} = \Gamma_{mn} \tag{C.5}$$

for some unique $j \in \{1, 2, \ldots, \ell_i\}$ and some unique $n \in \{1, 2, \ldots, \ell_m\}$.

Furthermore, given the relation (C.5) for some (unique) $j \in \{1, 2, \ldots, \ell_i\}$, some $m \in \mathcal{I}_i$, and some (unique) $n \in \{1, 2, \ldots, \ell_m\}$, we require $m \in \mathcal{I}_i$ is unique as well.

C3 For each $i = 1, 2, \ldots, k$, let

$$\mathcal{J}_i = \{j \in \{1, 2, \ldots, \ell_i\} : \mathbf{q}^i \circ \alpha_{ij}((a_{ij}, b_{ij})) \cap \mathbf{q}^m(\overline{V_m}) = \phi,\ m = 1, 2, \ldots, \ell_m\},$$

$$E_i = \bigcup_{j \in \mathcal{J}_i} \mathbf{q}^i(\Gamma_{ij})$$

and

$$E = \bigcup_{i=1}^{k} E_i.$$

We assume/require

$$\mathcal{S} = \left(\bigcup_{i=1}^{k} \mathbf{q}^i(\overline{V_i}) \right) \backslash E = \bigcup_{i=1}^{k} \left(\mathbf{q}^i(\overline{V_i}) \backslash E_i \right). \tag{C.6}$$

C4 For each $m \in \mathcal{I}_i$, $i = 1, 2, \ldots, k$ for which (C.5) holds we require

$$\gamma_{ij}(s) = \gamma_{ij}(a_{ij}) + s\ \gamma'_{ij}(a_{ij}) \qquad \text{for } 0 \le s \le b_{ij} - a_{ij}. \tag{C.7}$$

C5 As a consequence of (C.7) and the form of $\mathbf{q}^i$ given in (C.4) there holds

$$\mathbf{q}^i \circ \alpha_{ij}(s) = (A_i(\gamma_{ij}(a_{ij}))^T)^T + \mathbf{b}_i + s\ (A_i(\gamma'_{ij}(a_{ij}))^T)^T.$$

We require furthermore that for each $m \in \mathcal{I}_i$, $i = 1, 2, \ldots, k$ for which (C.5) holds, exactly one of the conditions

$$(A_i(\gamma_{ij}(a_{ij}))^T)^T + \mathbf{b}_i + s\ \mathbf{u}_{ij} = (A_m(\gamma_{mn}(a_{mn}))^T)^T + \mathbf{b}_m + s\ \mathbf{u}_{mn}$$

or

$$(A_i(\gamma_{ij}(a_{ij}))^T)^T + \mathbf{b}_i + s\ \mathbf{u}_i = (A_m(\gamma_{mn}(a_{mn}))^T)^T + \mathbf{b}_m - s\ \mathbf{u}_m$$

holds for $0 \leq s \leq c_{ij}$ where

$$\mathbf{u}_{ij} = \frac{(A_i[\alpha'_{ij}(a_{ij})]^T)^T}{\left|(A_i[\alpha'_{ij}(a_{ij})]^T)^T\right|}, \qquad \mathbf{u}_{mn} = \frac{(A_m[\alpha'_{mn}(a_{mn})]^T)^T}{|(A_m[\alpha'_{mn}(a_{mn})]^T)^T|},$$

and

$$c_{ij} = \left|(A_i[\alpha'_{ij}(a_{ij})]^T)^T\right| = |(A_m[\alpha'_{mn}(a_{mn})]^T)^T|. \tag{C.8}$$

Note: Since A_i has full rank (2) we know

$$c_{ij} = \left|(A_i[\alpha'_{ij}(a_{ij})]^T)^T\right| \neq 0.$$

Similarly,

$$c_{mn} = |(A_m[\alpha'_{mn}(a_{mn})]^T)^T| \neq 0.$$

The condition (C.8) requiring $c_{ij} = c_{mn}$ imposes an affine compatibility between $\mathbf{q}^i$ and $\mathbf{q}^m$ when $m \in \mathcal{I}_i$.

Given the face conditions **F1-F6** and the chart conditions **C1-C5** above and noting carefully the convention introduced in **C2** that

$$(\mathbf{q}^i)^{-1} = \left(\mathbf{q}^i\Big|_{\overline{V_i}}\right)^{-1} : \mathbf{q}^i(\overline{V_i}) \to \overline{V_i}.$$

as well as condition(C.6), we consider the set

$$V_0 = \bigcup_{i=1}^{k} (\mathbf{q}^i)^{-1}(\mathcal{S}).$$

If $j_0 \in \{1, 2, \ldots, \ell_i\} \backslash \mathcal{J}_i$, then there is some unique $m \in \mathcal{I}_i$ for which according to (C.5)

$$\mathbf{q}^i(\overline{V_i}) \cap \mathbf{q}^m(\overline{V_m}) = \Gamma_{ij_0} = \Gamma_{mn}.$$

By **C4-C5** exactly one of the conditions

$$\mathbf{q}^i \circ \alpha_{ij_0}(s) = \mathbf{q}^m \circ \alpha_{mn}(s)$$

or

$$\mathbf{q}^i \circ \alpha_{ij_0}(s) = \mathbf{q}^m \circ \alpha_{mn}(b_{mn} - a_{mn} - s)$$

holds for $0 \leq s \leq b_{mn} - a_{mn}$ where $b_{mn} - a_{mn} = b_{ij_0} - a_{ij_0}$.

Note carefully here the requirement of **C2**: For each $m \in \mathcal{I}_i$ it is required that a uniqe j exist satisfying (C.5). Given the relation (C.5), the index m is required to be unique. See Exercise C.10 below.

We define an auxiliary chart-like domain $\tilde{V}$ which is the topological quotient space obtained from the partition of V_0 constructed as follows:

(i) If $\mathbf{x} \in V_0 \cap V_i$, then we take the singleton $\{\mathbf{x}\}$ to be in $\tilde{V}$.

(ii) If $\mathbf{x} \in V_0 \cap \Gamma_{ij}$ and $\mathbf{q}^i(\mathbf{x}) = \mathbf{q}^m(\mathbf{y})$ for some $m \in \{1, 2, \dots, k\}\backslash\{i\}$ and some $\mathbf{y} \in \overline{V_m}$, then we take the set

$$\{\mathbf{y} \in V_0 : (\mathbf{q}^m)^{-1}(\mathbf{y}) = \mathbf{q}^i(\mathbf{x}) \ m \in \{1, 2, \dots, k\}\backslash\{i\}\}$$

to be in $\tilde{V}$.

Finally, we say $\mathcal{S}$ is a **piecewise affine surface** if the map $\tilde{\mathbf{q}} : \tilde{V} \to \mathcal{S}$ induced by the map $\mathbf{q} : V_0 \to \mathcal{S}$ given by

$$\mathbf{q}(\mathbf{x}) = \mathbf{q}^i(\mathbf{x}), \qquad \mathbf{x} \in V_i \text{ for } i = 1, 2, \dots, k$$

is a homeomorphism.

Exercise C.9. Show the trihedral corner $\mathcal{C}'$ is a piecewise affine surface. Hint: Figure C.7.

If the definitions of regular embedded surface and piecewise affine surface above are formulated correctly, then these should be examples of Poincaré manifolds.

Exercise C.10. Show that if we drop the additional condition concerning the uniqueness of m in the chart condition **C2** of Definition 24, then sets like the singular triple junction

$$\mathcal{T} = \left\{ t\left(\cos\frac{2j\pi}{3}, t\sin\frac{2j\pi}{3}, x_3\right) : 0 \le t < 1, -1 < z < 1, \ j = 0, 1, 2 \right\}$$

illustrated in Figure C.8 are allowed by/satisfy the resulting definition.

Let me finally return to Ruijia's original question. Given the usual use of the term "smooth structure," it is easy to put a smooth structure on the dihedral corner surface $\mathcal{C}'$. We have already done it. The only thing that

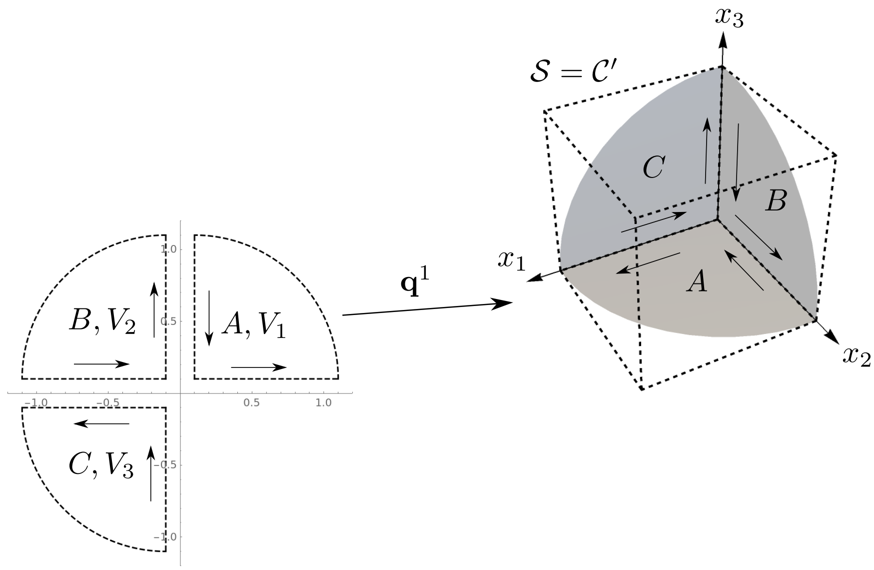

Figure C.7: Affine chart function $\mathbf{q}^1 : V_1 \to \mathcal{C}'$ for a piecewise affine embedded trihedral corner $\mathcal{S} = \mathcal{C}' \subset \mathbb{R}^3$.

is required is a single bijection $\mathbf{p} : U \to \mathcal{C}'$ where U is some open subset of $\mathbb{R}^2$. Such a bijection is illustrated on the right in Figure C.4. We do not even need continuity, so this assertion also is valid for the strange manifold $\mathcal{C}$ mentioned in Chapter 3.

What I have written in the previous paragraph may seem terribly mysterious, and in a sense it is. But what I have written is also correct. The key is in the meaning of "smooth structure." One way to think of this is as some formal notion according to which one can make a distinction among real valued functions $f : \mathcal{C}' \to \mathbb{R}$ concerning which ones have particular regularity properties.

Taking one step back, if we want a "topological structure" on a (point) set like $\mathcal{B}$ (the Riemannian manifold/disk of Chapter 3), $\mathcal{C}$ (the spray of points from Chapter 3 which will also be a Riemannian manifold when we get done with it), or $\mathcal{C}'$ (Ruijia's dihedral corner surface—a part of Ruijia's surface of a cube that I've isolated), then you need a topology. If you have a topology, you can distinguish the continuous real valued functions $f : \mathcal{C}' \to \mathbb{R}$ from the discontinuous ones. The set $\mathcal{C}'$ has an induced topology from $\mathbb{R}^3$, so this is one way to put a topology (or "topological structure") on $\mathcal{C}'$. But

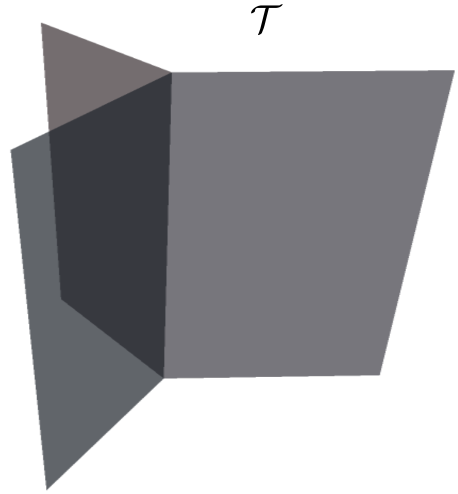

Figure C.8: A singlular triple junction. Such sets are interesting in other contexts, notably in geometric measure theory and modeling soap films, but they are rather complicated.

we can also say a set $U \subset \mathcal{C}'$ is open in $\mathcal{C}'$ if and only if $\mathbf{p}^{-1}(U)$ is open in $B_1(\mathbf{0})$. It turns out this gives you the same thing. That is similarly true for $\mathcal{B}$. It is not true for the spray of points $\mathcal{C}$. In that case, we ignore the subspace topology on $\mathcal{C}$ inherited from $\mathbb{R}^3$, and we specifically choose the topological structure/topology induced by the bijection $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}$ from Exercise 3.16: A set $U \subset \mathcal{C}$ is open in $\mathcal{C}$ if and only if $\mathbf{p}^{-1}(U) = \xi(U)$ is open in $B_1(\mathbf{0})$. That gives me a topology on that spray of points $\mathcal{C}$ and I can tell if a function $f : \mathcal{C} \to \mathbb{R}$ is continuous with respect to that topology. That is a topological structure.

Exercise C.11. Show a real valued function $f : \mathcal{C}' \to \mathbb{R}$ defined on the trihedral corner surface is continuous if and only if $f \circ \mathbf{p} \in C^0(B_1(\mathbf{0}))$. Similarly, given any topological space X, the function $f : \mathcal{C} \to X$ is continuous (with respect to the topology on $\mathcal{C}$ induced by the bijection $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}$) if and only if $f \circ \mathbf{p} \in C^0(B_1(\mathbf{0}) \to X)$. Finally, show $f : X \to \mathcal{C}$ is continuous if and only if $\xi \circ f \in C^0(X \to B_1(\mathbf{0}))$ where $\xi = \mathbf{p}^{-1} : \mathcal{C} \to B_1(\mathbf{0})$.

In much the same way, the continuous bijections $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{B}$, $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}$ and $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}'$ allow me to determine if a real valued function f with domain one of these spaces is differentiable, C^1, C^2, etc., or C^∞. How do I do that? Here is how:

Remember this is (or these are) just definitions:

A real valued function $f : \mathcal{C}' \to \mathbb{R}$ is **differentiable** if $f \circ \mathbf{p} : B_1(\mathbf{0}) \to \mathbb{R}$ is differentiable.

This definition has the advantage that it certainly distinguishes some functions $f : \mathcal{C}' \to \mathbb{R}$ as "differentiable" and others as not differentiable. This definition also has the irritating, and probabably justifiably objectionable, characteristic that there is no function $g : \mathcal{C}' \to \mathbb{R}$ which can be recognized as a partial derivative (or more generally as a directional derivative) of a "differentiable" function f. This can be understood in a couple different ways. First of all, partial derivatives of the composition $f \circ \mathbf{p}$ would, of course, not really qualify as partial derivatives of the function f itself. We will see this more clearly when we consider additional chart functions, but it is pretty obvious that if there is a notion of a partial derivative of f, then a partial derivative of some composition involving f like $f \circ \mathbf{p}$ should not be it. Furthermore, there is no way to directly calculate a partial derivative of $f : \mathcal{C} \to \mathbb{R}$. Of course, in some instances there is a way you could try. For the Riemannian manifold $\mathcal{B}$, you could happily take (partial) difference quotients with respect to the Euclidean coordinates in $B_1(\mathbf{0})$ since this is the "same" as $\mathcal{B}$ as a point set, but if you think about it, you will get the wrong answer because the Euclidean distance in $B_1(\mathbf{0})$ is not correct with respect to the distances in $\mathcal{B}$. We can see this more clearly in the next section below.

Let's pause for a moment to consider carefully the difference between this "new" definition of differentiability and a/the more familiar one. We say a function $f : U \to \mathbb{R}$ defined on an open set $U \subset \mathbb{R}^2$ is **differentiable** at $\mathbf{x} \in U$ if there exists a linear function $L : \mathbb{R}^2 \to \mathbb{R}^1$ such that

$$\lim_{\mathbf{p} \to \mathbf{x}} \frac{f(\mathbf{p}) - f(\mathbf{x}) - L(\mathbf{x} - \mathbf{p})}{|\mathbf{x} - \mathbf{p}|} = 0.$$

In this case the linear function $L \in \beth(\mathbb{R}^2 \to \mathbb{R})$ is called the **differential** of f at $\mathbf{x}$ and is denoted by $df_{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}$. Furthermore, the quantities

$$\lim_{v \to 0} \frac{f(\mathbf{x} + v\mathbf{e}_j) - f(\mathbf{x})}{v}$$

are well-defined for $j = 1, 2$ where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. These are called **partial derivatives** and are denoted by

$$\frac{\partial f}{\partial x_j}(\mathbf{x}) = \lim_{v \to 0} \frac{f(\mathbf{x} + v\mathbf{e}_j) - f(\mathbf{x})}{v}.$$

More generally, differentiability at a point also implies the existence of the limits

$$D_{\mathbf{u}}f(\mathbf{x}) = \lim_{v\to 0} \frac{f(\mathbf{x}+v\mathbf{u}) - f(\mathbf{x})}{v} \tag{C.9}$$

for any $\mathbf{u} \in \mathbb{R}^2$, and these limits are called **directional derivatives**. Sometimes the notation and limits taken in (C.9) is limited to situations in which $\mathbf{u} \in \mathbb{S}^1$ (or more generally in which $|\mathbf{u}| = 1$) to emphasize that the value gives the rate of change of the function f in a certain direction.

Exercise C.12. Give an interpretation of $D_{\mathbf{u}}f(\mathbf{x})$ as a rate of change of values of the function f when $\mathbf{u}$ is not a unit vector.

We say f is **differentiable on all of** U if f is differentiable at each point $\mathbf{x} \in U$. In this case, the differential $df_{\mathbf{x}}$ can be used to define a function $pf : U \times \mathbb{R}^2 \to \mathbb{R}$ by $pf(\mathbf{x}, \mathbf{v}) = df_{\mathbf{x}}(\mathbf{v})$. The function pf, which we can call the **point differential function** has various properties. For example, pf is linear in the second argument. We do not know much, however, about the regularity of pf in the first argument. In this case, there is also a natural function assigning to each $\mathbf{x} \in U$ a (bounded) linear functional in $\daleth(\mathbb{R}^2)$. This function is naturally denoted $df : U \to \daleth(\mathbb{R}^2)$ and is also called the **differential**. We can distinguish between $df_{\mathbf{x}}$ and df by calling the former the **differential at a point** and the latter the **differential function** or **differential form** if we like. Finally, in this case, the values of the partial derivatives define functions

$$\frac{\partial f}{\partial x_j} : U \to \mathbb{R}$$

for $j = 1, 2$. Again, we do not know anything about the regularity of these functions. In summary, the above is differentiability for a real valued function f defined on an open subset of $\mathbb{R}^2$.

Now we can add a couple more observations about differentiability we have left out. These have to do with collections of functions satisfying the conditions of differentiability.

1. If $f : U \to \mathbb{R}$ is a function which is differentiable at a point $\mathbf{x} \in U$, then the function $cf : U \to \mathbb{R}$ obtained by scaling the value of f by a constant $c \in \mathbb{R}$ is also differentiable at $\mathbf{x}$.

2. If $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are two functions both of which are differentiable at a point $\mathbf{x} \in U$, then the $f + g : U \to \mathbb{R}$ obtained by adding the values of f and g is also differentiable at $\mathbf{x}$.

3. Consequently the collection of all functions $f : U \to \mathbb{R}$ which are differentiable at the point $\mathbf{x} \in U$ is a real linear space.[7] This linear space is not used so often, and I do not know that it has any special name or notation associated with it. It is a well-defined linear space all the same.

4. The collection of all differentiable real valued functions on all of U is also a linear space. This real linear space also does not seem to have a (standard) notation, but it deserves one. Here is my suggestion: $\mathrm{Diff}^1(U)$.

5. If you think about it, there are a number of other natural linear spaces to consider in connection with $\mathrm{Diff}^1(U)$. For example, the collection of differential functions associated with differentiable functions in $\mathrm{Diff}^1(U)$ is a linear space. That is,

$$\mathrm{Dform}^1(U) = \{df \in \beth(\mathbb{R}^2)^U : f \in \mathrm{Diff}^1(U)\}.$$

Exercise C.13. Verify that the collection of all point differential functions $pf : U \times \mathbb{R}^2 \to \mathbb{R}$ associated with functions $f \in \mathrm{Diff}^1(U)$ is a real linear space, and make up a good notation for this **space of point differential functions**.

Exercise C.14. Given an open set $U \subset \mathbb{R}^2$, the collection of all functions $\mathbf{v} : U \to \mathbb{R}^2$, that is, $(\mathbb{R}^2)^U$, is a linear space. You can check it. Verify that the collection of all functions $vf : (\mathbb{R}^2)^U \to \mathbb{R}^U$ by $vf(\mathbf{v})(\mathbf{x}) = pf(\mathbf{x}, \mathbf{v}(\mathbf{x}))$ for some $f \in \mathrm{Diff}^1(U)$ is a real linear space. Of course, the space $(\mathbb{R}^2)^U$ is the space of **vector fields** on U. The function vf determined by the differential of a function $f \in \mathrm{Diff}^1(U)$ may be called a **point-field differential function**.

In contrast, we have a new definition of differentiability for sets like $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{C}'$. This new definition uses the old definition above. Take for example Ruijia's trihedral corner surface $\mathcal{C}'$:

> A function $f : \mathcal{C}' \to \mathbb{R}$ is **differentiable** if $f \circ \mathbf{p} : B_1(\mathbf{0}) \to \mathbb{R}$ is differentiable (in the old sense).

[7] You may note that I am trying to use the term "linear space" here in contrast to the usual "vector space." This is not by accident, and the rationale for such, perhaps inconvenient and perhaps subversive, distinctions will be elaborated later.

Let's assume for a moment that this new definition of differentiability is reasonable. What does it get us? As mentioned above, it does not get us a derivative (or partial derivatives or directional derivatives) for f. Let us leave aside for a moment the question of whether or not we can get a reasonable notion of a differential for f or any of the various flavors of differential functions discussed above. I will have a good deal to say about that later. What we can say for sure is that this new (relatively simple) notion of differentiable does get us some linear spaces. The collection of all real valued differentiable functions, according to this new definition, is a linear space for sure. To see this, simply note that if $f : \mathcal{C}' \to \mathbb{R}$ satisfies

$$f \circ \mathbf{p} \in \text{Diff}^1(B_1(\mathbf{0}))$$

and $c \in \mathbb{R}$, then (first of all) cf is a well-defined real valued function on the trihedral corner surface $\mathcal{C}'$. Furthermore $(cf) \circ \mathbf{p}$ has the same values as $c(f \circ \mathbf{p})$. Since $\text{Diff}^1(B_1(\mathbf{0}))$ is a linear space, we know then

$$c(f \circ \mathbf{p}) \in \text{Diff}^1(B_1(\mathbf{0})).$$

And it follows that cf satisfies the new definition of differentiability on $\mathcal{C}'$. Similar reasoning applies to $f + g$ when f, g are (new) differentiable functions on $\mathcal{C}'$.

This observation about the linear space of (new) differentiable functions on a set like the trihedral corner surface $\mathcal{C}'$ is largely what is meant—in its entirety—by saying $\mathcal{C}'$ has a "smooth structure" or is given a smooth structure by the chart function $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}'$.

Now let's take a moment to be a little more critical. Is it really reasonable to say a function $f : \mathcal{C}' \to \mathbb{R}$ is **differentiable** just because $f \circ \mathbf{p} : B_1(\mathbf{0}) \to \mathbb{R}$ is differentiable (in the old sense)? Of course, it does say something about the function f as compared to other real valued functions with domain $\mathcal{C}'$ which might not satisfy this condition. On the other hand, we do not even get a partial derivative or directional derivatives out of the deal. From this point of view, I think it is reasonable to say this property should actually not be called differentiability. It should be called something else. And I have a suggestion. We could do this:

> If a function $f : \mathcal{C}' \to \mathbb{R}$ has the property that $f \circ \mathbf{p} \in \text{Diff}^1(B_1(\mathbf{0}))$, then we can say the function f is **chart differentiable**.[8]

[8]. . . or differentiable with respect to the chart—or even **differentiable with respect to charts** if we have more than one chart, which is a situation we have not really addressed in any substantial way yet.

I actually like this terminology and the idea behind it quite a lot. But you must realize that if you go with me on this, you will be bucking the trend that I guess started with Hermann Weyl who is one of the recognized founders of the subject of Riemannian geometry and is recognized (along with a lot of other people who are not me) as something of an authority. On the other hand, I'm no fan of authoritah, so I'm happy to go further in my blasphemy.

One can ask the question:

> Should the fact that I have a bijection between a nice respectable open set like $B_1(\mathbf{0}) \subset \mathbb{R}^2$ and some point set $\mathcal{C}$ be conflated to justify the terminology that I've put a "smooth structure" on $\mathcal{C}$?

Who is responsible for this terminology anyway? Is there a justification for it? If there is, I suppose it is merely the fact that it allows the consideration of various linear spaces as discussed above. If we wish to make a critical response, as with differentiability we can offer a new name/term. I suggest that instead of "smooth structure," we might say topological spaces like $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{C}'$ are given a **chart structure**.

We have used the chart structure to actually get a topology on the trihedral corner surface $\mathcal{C}'$, and then given any topological space X, it makes good sense to consider the continuous functions $C^0(X \to \mathcal{C}')$ or the continuous functions $C^0(\mathcal{C}' \to X)$. These are not linear spaces. The real valued continuous functions, however, $C^0(\mathcal{C}')$ and/or the continuous functions $C^0(\mathcal{C}' \to \mathbb{R}^n)$, are linear spaces.

We can use the chart structure given by $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}'$ to also define the following linear spaces

$$\{f \in C^0(\mathcal{C}') : f \circ \mathbf{p} \in \mathrm{Diff}^1(B_1(\mathbf{0}))\}$$

$$\{f \in C^0(\mathcal{C}') : f \circ \mathbf{p} \in C^1(B_1(\mathbf{0}))\} \tag{C.10}$$

$$\{f \in C^0(\mathcal{C}') : f \circ \mathbf{p} \in C^2(B_1(\mathbf{0}))\} \tag{C.11}$$

$$\vdots$$

$$\{f \in C^0(\mathcal{C}') : f \circ \mathbf{p} \in C^\infty(B_1(\mathbf{0}))\}. \tag{C.12}$$

We have seen above that

$$f \in C^0(\mathcal{C}') \qquad \iff \qquad f \circ \mathbf{p} \in C^0(B_1(\mathbf{0})).$$

See Exercise C.11. Recall the linear space $C^1(B_1(\mathbf{0}))$ consists of those functions $f \in \text{Diff}^1(B_1(\mathbf{0}))$ for which each of the partial derivatives

$$\frac{\partial f}{\partial x_j}$$

satisfies

$$\frac{\partial f}{\partial x_j} \in C^0(B_1(\mathbf{0})).$$

Consequently, the set in (C.10) is a linear space. This linear space might usually be referred to as $C^1(\mathcal{C}')$ and the elements called continuously differentiable functions on $\mathcal{C}'$. I think it's fair, however, to object to such terminology and notation, but to be constructive we might wish to offer a replacement. As we had chart differentiable functions $f \in C^0(\mathcal{C}')$, we could designate the linear space in (C.10) as the collection of **chart C^1 functions** and write

$$\text{cC}^1(\mathcal{C}') = \{f \in C^0(\mathcal{C}') : f \circ \mathbf{p} \in C^1(B_1(\mathbf{0}))\}.$$

Exercise C.15. The definition I've given above for $C^1(B_1(\mathbf{0}))$ is not quite the usual one. More generally, if U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$, then f is said to be **partially differentiable** at $\mathbf{x} \in U$ if for each $j = 1, 2, \ldots, n$ the limit

$$\lim_{v \to 0} \frac{f(\mathbf{x} + v\mathbf{e}_j) - f(\mathbf{x})}{v} \tag{C.13}$$

exists. Also when the limit in (C.13) exists, the value of the limit is denoted of course by

$$\frac{\partial f}{\partial x_j}(\mathbf{x}) = \lim_{v \to 0} \frac{f(\mathbf{x} + v\mathbf{e}_j) - f(\mathbf{x})}{v}.$$

As usual f is said to be **partially differentiable on all of** U if f is partially differentiable at each point, and in this case the values of the partial derivatives define functions

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \in \mathbb{R}^U.$$

The collection (which is a linear space) of partially differentiable functions may be denoted by pDiff(U). The usual definition of $C^1(U)$ is

$$C^1(U) = \left\{ f \in \text{pDiff}(U) : \frac{\partial f}{\partial x_j} \in C^0(U) \text{ for } j = 1, 2, \ldots, n \right\}. \tag{C.14}$$

Show this definition is equivalent to the nonstandard one given above. That is, show that each function $f \in C^1(U)$ according to (C.14) is differentiable, i.e., satisfies $f \in \text{Diff}^1(U)$.

Exercise C.16. Show $\text{cC}^1(\mathcal{C}') \subset C^0(\mathcal{C}')$.

I'll finish up this section by tidying up and summarizing the proposed notation:

$$\begin{aligned}
\text{cDiff}^1(\mathcal{B}) &= \{f \in C^0(\mathcal{B}) : f \circ \mathbf{p} \in \text{Diff}^1(B_1(\mathbf{0}))\} \\
\text{cC}^1(\mathcal{B}) &= \{f \in C^0(\mathcal{B}) : f \circ \mathbf{p} \in C^1(B_1(\mathbf{0}))\} \\
\text{cC}^2(\mathcal{B}) &= \{f \in C^0(\mathcal{B}) : f \circ \mathbf{p} \in C^2(B_1(\mathbf{0}))\} \\
&\vdots \\
\text{cC}^\infty(\mathcal{B}) &= \{f \in C^0(\mathcal{B}) : f \circ \mathbf{p} \in C^\infty(B_1(\mathbf{0}))\}.
\end{aligned}$$

These are the **chart differentiable functions on** $\mathcal{B}$, the **chart** C^1 **functions on** $\mathcal{B}$, the **chart** C^2 **functions on** $\mathcal{B}$, and so on, to the **chart** C^∞ **functions on** $\mathcal{B}$.

C.6 Lance's idea/answer

It can also be good to ask a well-posed question. This can especially be the case when the answer to the question is not immediately obvious (to you) but you think you have a good chance to answer the question. Though I may not have posed the question, originally finding its origin in the ideas of Travis and Ruijia above, using the discussion of smooth embedded regularly parameterized surfaces included in my comments about Ruijia's question above, we now have the means to pose it precisely:

> Does there exist a surface $\mathcal{S} \subset \mathbb{R}^3$ regularly embedded by some single chart function $\mathbf{q} : B_1(\mathbf{0}) \to \mathbb{R}^3$ where $B_1(\mathbf{0}) \subset \mathbb{R}^2$ so that the natural bijection $\psi : \mathcal{B} \to \mathcal{S}$ by $\psi(P) = \mathbf{q} \circ (\mathbf{x})$ where $\mathbf{p}(\mathbf{x}) = P \in \mathcal{B}$ is the point set identity has the property that paths $\alpha \in C^1([a, b] \to B_1(\mathbf{0}))$ have corresponding paths
>
> $$\beta = \mathbf{q} \circ \alpha \in C^1([a, b] \to \mathcal{S})$$
>
> with length given precisely by
>
> $$\text{length}_{\mathcal{S}}[\beta] = \text{length}_{\mathcal{B}}[\alpha] = \int_{(a,b)} \frac{4}{4 + |\alpha|^2} \, |\alpha'|?$$

The answer to this question is "yes," and I think Lance Lampert has found such a surface and a chart function to go along with it. I will leave it to Lance to explain his answer.

It occurs to me, however, that it may not be entirely clear to some of you how the verification of the assertion would go. . . even if you had the surface and the chart in hand. Thus, I will try to briefly explain that here.

How to compute the lengths of curves on a surface

Say we have a chart $\mathbf{q} : B_1(\mathbf{0}) \to \mathbb{R}^3$ whose image is a regular embedded surface $\mathcal{S}$. In particular, we assume $\mathbf{q}$ is a regular embedding, so the vectors

$$\frac{\partial \mathbf{q}}{\partial x_1} \qquad \text{and} \qquad \frac{\partial \mathbf{q}}{\partial x_2}$$

are linearly independent. And I assume you can compute these vectors. When Lance gives his answer, I guess you will be eager to do so. Thus, you have two functions

$$\frac{\partial \mathbf{q}}{\partial x_j} : B_1(\mathbf{0}) \to \mathbb{R}^3$$

for $j = 1, 2$. Now, say you have a curve $\alpha : [a, b] \to \mathcal{S} \subset \mathbb{R}^3$. In fact, it makes perfectly good sense, since each coordinate function α^j of $\alpha = (\alpha^1, \alpha^2, \alpha^3)$ is a real valued function on $B_1(\mathbf{0})$, to assume $\alpha \in C^1([a, b] \to \mathbb{R}^3)$. In this case, we could also use "chart regularity" as described above considering $\mathcal{S}$ as a subset of $\mathbb{R}^3$ with chart structure, a.k.a. smooth structure, induced by the chart $\mathbf{q}$. In partiucular to say $\alpha \in \mathrm{cC}^1([a, b] \to \mathcal{S})$ would mean the composition

$$\mathbf{q}^{-1} \circ \alpha \in C^1([a, b] \to B_1(\mathbf{0})).$$

Let us denote $\mathbf{q}^{-1}$ by $\eta : \mathcal{S} \to B_1(\mathbf{0})$. Then $\beta = \eta \circ \alpha$ gives the path in $B_1(\mathbf{0})$ corresponding to α. Naturally, we can think of the image of β as being in $\mathcal{B}$, so we can calcualate the Riemannian length $\text{length}_{\mathcal{B}}[\beta]$. And the question is:

Is $\text{length}_{\mathcal{B}}[\beta] = \text{length}_{\mathcal{B}}[\eta \circ \alpha]$ always the same as $\text{length}_{\mathcal{S}}[\alpha] = \text{length}_{\mathbb{R}^3}[\alpha]$?

In order to check this, since we know $\text{length}_{\mathcal{B}}[\beta]$ is given by

$$\text{length}_{\mathcal{B}}[\beta] = \int_{(a,b)} \frac{4}{4 + |\beta|^2} \, |\beta'|,$$

it makes sense to reexpress α as $\alpha = \mathbf{q} \circ \beta$. Then

$$\text{length}_{\mathbb{R}^3}[\alpha] = \int_{(a,b)} |\alpha'| = \int_{(a,b)} |(\mathbf{q} \circ \beta)'|.$$

Thus, the question becomes: What is $(\mathbf{q} \circ \beta)'$? Well, remember the function $\beta = (\beta^1, \beta^2)$ has two component functions since $\beta : [a,b] \to B_1(\mathbf{0}) \subset \mathbb{R}^2$. Thus, using the chain rule

$$\begin{aligned}\frac{d}{dt}(\mathbf{q} \circ \beta) &= \frac{\partial \mathbf{q}}{\partial x_1}(\beta)\ \frac{d\beta^1}{dt} + \frac{\partial \mathbf{q}}{\partial x_2}(\beta)\ \frac{d\beta^2}{dt} \\ &= \frac{\partial \mathbf{q}}{\partial x_1}\ \frac{d\beta^1}{dt} + \frac{\partial \mathbf{q}}{\partial x_2}\ \frac{d\beta^2}{dt}.\end{aligned}$$

Therefore,

$$\begin{aligned}|(\mathbf{q} \circ \beta)'|^2 &= \left\langle \frac{\partial \mathbf{q}}{\partial x_1}\ \frac{d\beta^1}{dt} + \frac{\partial \mathbf{q}}{\partial x_2}\ \frac{d\beta^2}{dt}, \frac{\partial \mathbf{q}}{\partial x_1}\ \frac{d\beta^1}{dt} + \frac{\partial \mathbf{q}}{\partial x_2}\ \frac{d\beta^2}{dt} \right\rangle_{\mathbb{R}^3} \\ &= \left\langle \frac{\partial \mathbf{q}}{\partial x_1}(\beta), \frac{\partial \mathbf{q}}{\partial x_1}(\beta) \right\rangle \left(\frac{d\beta^1}{dt} \right)^2 \\ &\qquad + 2 \left\langle \frac{\partial \mathbf{q}}{\partial x_1}(\beta), \frac{\partial \mathbf{q}}{\partial x_2}(\beta) \right\rangle \frac{d\beta^1}{dt}\ \frac{d\beta^2}{dt} \\ &\qquad\qquad + \left\langle \frac{\partial \mathbf{q}}{\partial x_2}(\beta), \frac{\partial \mathbf{q}}{\partial x_2}(\beta) \right\rangle \left(\frac{d\beta^2}{dt} \right)^2.\end{aligned}$$

On the face of it, it may seem rather unlikely that this quantity will turn out to satisfy

$$|(\mathbf{q} \circ \beta)'|^2 = \left(\frac{4}{4 + |\beta|^2}\ |\beta'| \right)^2 = \frac{16}{(4 + |\beta|^2)^2}\ \left[\left(\frac{d\beta^1}{dt} \right)^2 + \left(\frac{d\beta^2}{dt} \right)^2 \right],$$

but that is the calculation to be checked.

We haven't really considered multiple charts, and I do not have time at the moment to type up a detailed explanation of how that works, but we should be doing that soon, and when we do, the following question will be natural. In some sense it is natural now, and you should be able to do it if Lance's answer is correct (and understood), so I'm going to go ahead and record the exercise. Then I'll make some pictures and provide some explanation to go along with it later.

Exercise C.17. While it is possible to give a global chart function for the trihedral corner surface $\mathcal{C}'$ considered above, it is not possible to give a global chart function for the entire surface $\partial C_1(\mathbf{0})$ of a cube in $\mathbb{R}^3$ that was mentioned by Ruijia. Notice there are eight vertices on the surface of the cube.

Give eight chart functions, one for each vertex of the cube that induce on the entire surface of the cube a chart structure (a.k.a., a smooth structure) which is isometric to the natural chart structure on the extension of $\mathcal{B}$ to a compact Riemannian manifold.