D.2 Chapter 16

D.2.1 Local chart formula for the gradient

Solution of Exercise. (Exercise 16.12, page 16.12)

(a) Here we have a vector $w \in T_P M$ for which

$$\mu_P(w, u) = D_u f(P) \qquad \text{for every } u \in \mathbb{S}_P^{n-1} \subset T_P M.$$

Define two linear functions $L_j : T_P M \to \mathbb{R}$ for $j = 1, 2$ by

$$L_1(v) = \mu_P(w, v),$$
$$L_2(v) = df_P(v)$$

respectively. Recall that $df_P(u) = D_u f(P)$ gives the directional derivative of f when $u \in \mathbb{S}_P^{n-1} \subset T_P M$. Thus, we have $L_1(u) = L_2(u)$ for $u \in \mathbb{S}_P^{n-1}$. If $v \in T_P M \backslash \{\mathbf{0}\}$, then

$$L_1(v) = \|v\|_{T_P M} L_1(u) = \|v\|_{T_P M} L_2(u) = L_v(v)$$

where $u = v/\|v\|_{T_P M}$. The only other element of $T_P M$ is $\mathbf{0} \in T_P M$, and we know simply because L_1 and L_2 are linear that $L_1(\mathbf{0}) = L_2(\mathbf{0})$. Thus, $L_1 \equiv L_2$, that is

$$\mu_P(w, v) = df_P(v) \qquad \text{for} \qquad v \in T_P M.$$

(b) If there are two vectors w and $\tilde{w}$ in $T_P M$ for which the assertion of part **(a)** holds, then $\mu_P(w, v) = \mu_P(\tilde{w}, v)$ for all $v \in T_P M$. This means

$$\mu_P(\tilde{w} - w, v) = 0 \qquad \text{for all } v \in T_P M.$$

In particular, taking $v = \tilde{w} - w$ we have

$$\|\tilde{w} - w\|_{T_P M} = \mu_P(\tilde{w} - w, \tilde{w} - w)^{1/2} = 0,$$

so $\tilde{w} - w = \mathbf{0}$ because the Riemannian inner product $\mu_P : T_P M \times T_P M \to \mathbb{R}$ is positive definite. That is, $\tilde{w} = w$ is unique.

Solution of Exercise. (Exercise 16.13, page 16.13) This exercise, if carried out as suggested, not only gives existence for the gradient (at a point) but also a formula in terms of a local chart.

(a) The first part here is to show $\{v_1, v_2, \ldots, v_n\}$ is a basis for $T_P M$ where $v_j = d\mathbf{p}_\mathbf{x}(\mathbf{e}_j)$ for $\mathbf{x} = \xi(P)$ and $j = 1, 2, \ldots, n$. In order to see that $\{v_1, v_2, \ldots, v_n\}$ spans $T_P M$, let z be any element of $T_P M$ and recall that $d\xi_P : T_P M \to T_\mathbf{x}\mathbb{R}^n$ is a linear isomorphism. Therefore,

$$d\xi_P(z) = \sum_{j=1}^{n} \langle d\xi_P(z), \mathbf{e}_j \rangle_{\mathbb{R}^n} \ \mathbf{e}_j = \sum_{j=1}^{n} \langle d\xi_P(z), \mathbf{e}_j \rangle_{\mathbb{R}^n} \ d\xi_P(v_j).$$

This is mostly just because $d\xi_P(z) \in T_\mathbf{x}\mathbb{R}^n$ and $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard orthonormal basis for $T_\mathbf{x}\mathbb{R}^n$. We have also used $v_j = d\mathbf{p}_\mathbf{x}(\mathbf{e}_j)$ for the replacement $\mathbf{e}_j = d\xi_P(v_j)$, $j = 1, 2, \ldots, n$.

Applying the linear inverse $d\mathbf{p}_\mathbf{x} : T_\mathbf{x}\mathbb{R}^n \to T_P M$ to both sides, we have

$$z = \sum_{j=1}^{n} \langle d\xi_P(z), \mathbf{e}_j \rangle_{\mathbb{R}^n} \ v_j = \sum_{j=1}^{n} c_j \ v_j$$

where $c_j = \langle d\xi_P(z), \mathbf{e}_j \rangle_{\mathbb{R}^n}$ for $j = 1, 2, \ldots, n$. This shows $\{v_1, v_2, \ldots, v_n\}$ is a spaning set for $T_P M$. It remains to show $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set.

If there are real numbers $c_1, c_2, \ldots, c_n$ for which

$$\sum_{j=1}^{n} c_j \ v_j = \mathbf{0} \in T_P M,$$

then applying $d\xi_P$ to both sides we find

$$\sum_{j=1}^{n} c_j \ \mathbf{e}_j = \mathbf{0} \in T_\mathbf{x}\mathbb{R}^n.$$

This implies $c_1 = c_2 = \cdots = c_n = 0$ and $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set.

(b) Now let us assume a gradient vector $Df(P) \in T_P M$ exists and write

$$Df(P) = \sum_{j=1}^{n} c_j v_j.$$

Then assuming

$$\mu_p(Df(P), z) = df_P(z) \qquad \text{for} \qquad z \in T_P M$$

as shown in part **(a)** of Exercise 16.12 we should have

$$\sum_{j=1}^{n} c_j \mu_P(v_j, z) = df_P(z). \tag{D.30}$$

Taking $z = v_i$ for $i = 1, 2, \ldots, n$, the numbers

$$df_P(v_i) = \frac{d}{dt}(f \circ \mathbf{p} \circ \gamma_i)(t)\Big|_{t=0} = \langle D(f \circ \mathbf{p})(\mathbf{x}), \mathbf{e}_i \rangle_{\mathbb{R}^n}$$

where $\gamma_i(t) = \mathbf{x} + t\mathbf{e}_i$ for $i = 1, 2, \ldots, n$ may be considered known. Thus, from (D.30) we obtain a system of n equations

$$\sum_{j=1}^{n} g_{ij} c_j = \langle D(f \circ \mathbf{p})(\mathbf{x}), \mathbf{e}_i \rangle_{\mathbb{R}^n} \qquad i = 1, 2, \ldots, n \tag{D.31}$$

for the coefficients $c_1, c_2, \ldots, c_n$. Writing $c = (c_1, c_2, \ldots, c_n)$, we can write these n linear equations as a single vector equation

$$(g_{ij})c^T = D(f \circ \mathbf{p})(\mathbf{x})^T$$

where we have recognized the left side of (D.31) as the inner product $\langle (g_{ij}), c \rangle_{\mathbb{R}^n}$ where $g_{ij}(\mathbf{x}) = \mu_P(v_i, v_j)$ and $(g_{ij}) = (g_{ij}(\mathbf{x}))$ is the matrix of metric coefficients in U, and we have recognized the right side of (D.31) as the i-th entry in the vector $D(f \circ \mathbf{p})(\mathbf{x}) \in T_{\mathbf{x}}\mathbb{R}^n$. Since the matrix of metric coefficients is invertible, if we write $(g^{ij}) = (g_{ij})^{-1}$, then the coefficients are given by

$$(c_1, c_2, \ldots, c_n) = c = [(g^{ij}) D(f \circ \mathbf{p})(\mathbf{x})^T]^T = D(f \circ \mathbf{p})(\mathbf{x})\ (g^{ij}). \tag{D.32}$$

This formal calculation tells us that if $Df(P)$ exists, then we should have

$$Df(P) = \sum_{j=1}^{n} c_j v_j \tag{D.33}$$

with coefficients given by (D.32).

(c) Given a chart $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$ with $P \in \mathbf{p}(U)$, we have a basis $\{v_1, v_2, \ldots, v_n\}$ for $T_P M$, and there is a vector

$$w = \sum_{j=1}^{n} c_j v_j \in T_P M$$

with coefficients $c_1, c_2, \ldots, c_n$ given by (D.32). Given $u \in \mathbb{S}_P^{n-1} \subset T_P M$, we make a calculation denoting the metric coefficient matirx by (g_{ij}) by G:

$$\begin{aligned}
\mu_P(w,u) &= \mu_P\left(\sum_{j=1}^{n} c_j v_j, u\right) \\
&= \left\langle G\left(\sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial(f\circ\mathbf{p})}{\partial x_j}(\mathbf{x})\; g^{ij} d\xi_P(v_j)\right), d\xi_P(u)\right\rangle_{\mathbb{R}^n} \\
&= \left\langle G\left(\sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}\frac{\partial(f\circ\mathbf{p})}{\partial x_i}(\mathbf{x})\; \mathbf{e}_j\right), d\xi_P(u)\right\rangle_{\mathbb{R}^n} \\
&= \left\langle G\left(\sum_{i=1}^{n} \frac{\partial(f\circ\mathbf{p})}{\partial x_i}(\mathbf{x}) \sum_{j=1}^{n} g^{ij}\; \mathbf{e}_j\right), d\xi_P(u)\right\rangle_{\mathbb{R}^n} \\
&= \left\langle G\left(\sum_{i=1}^{n} \frac{\partial(f\circ\mathbf{p})}{\partial x_i}(\mathbf{x}) g^{ij}\right), d\xi_P(u)\right\rangle_{\mathbb{R}^n} \\
&= \left\langle G\left(G^{-1}D(f\circ\mathbf{p})(\mathbf{x})^T\right), d\xi_P(u)\right\rangle_{\mathbb{R}^n} \\
&= \langle D(f\circ\mathbf{p})(\mathbf{x}), d\xi_P(u)\rangle_{\mathbb{R}^n}.
\end{aligned}$$

On the other hand, recall that $u = [\alpha]$ for some $\alpha : I \to \mathbf{p}(U)$ with $\alpha(t_0) = P$. Thus, we can find a path $\beta : I \to U$ with $\beta = \xi \circ \alpha$ and $d\xi_P(u) = \beta'(t_0)$. Therefore,

$$\begin{aligned}
D_u f(P) &= \lim_{t\to t_0} \frac{f\circ\alpha(t) - f(P)}{\int_{t_0}^{t} \langle G\beta', \beta'\rangle_{\mathbb{R}^n}^{1/2}\, d\tau} \\
&= \lim_{t\to t_0} \frac{f\circ\alpha(t) - f(P)}{t - t_0}\; \frac{t - t_0}{\int_{t_0}^{t} \langle G\beta', \beta'\rangle_{\mathbb{R}^n}^{1/2}\, d\tau} \\
&= (f\circ\alpha)'(t_0) \lim_{t\to t_0} \frac{t - t_0}{\int_{t_0}^{t} \langle G\beta', \beta'\rangle_{\mathbb{R}^n}^{1/2}\, d\tau}
\end{aligned}$$

where $G\beta' = (g_{ij}(\beta(\tau)))\beta'(\tau)^T$ and

$$\begin{aligned}\lim_{t\to t_0}\frac{1}{t-t_0}\int_{t_0}^{t}\langle G\beta',\beta'\rangle_{\mathbb{R}^n}^{1/2}\,d\tau &= \langle (g_{ij}(\mathbf{x}))\beta'(t_0),\beta'(t_0)\rangle_{\mathbb{R}^n}^{1/2}\\ &= \mu_P([\alpha],[\alpha])^{1/2}\\ &= \mu_P(u,u)^{1/2}\\ &= 1.\end{aligned}$$

We conclude

$$\begin{aligned}D_u f(P) &= (f\circ\alpha)'(t_0)\\ &= (f\circ\mathbf{p}\circ\beta)'(t_0)\\ &= \langle D(f\circ\mathbf{p})(\mathbf{x}),\beta'(t_0)\rangle_{\mathbb{R}^n}\\ &= \langle D(f\circ\mathbf{p})(\mathbf{x}),d\xi_P(u)\rangle_{\mathbb{R}^n},\end{aligned}$$

and

$$\mu_P(w,u) = D_u f(P) \qquad \text{for all } u\in\mathbb{S}_P^{n-1}.$$

This means the vector

$$w = \sum_{j=1}^{n} c_j v_j \in T_P M$$

with coefficients $c_1, c_2, \ldots, c_n$ given by (D.32) satisfies the basic requirement to be the gradient of f and in particular the hypotheses of Exercise 16.12. We have now shown that the gradient vector exists, and we have a formula for it in terms of a chart function $\mathbf{p} : U \to M$ with $P \in \mathbf{p}(U)$.

One could also show directly that the formula for $Df(P)$ given here is independent of the chart $(U,\mathbf{p}) \in \mathcal{A}_*^\infty$. This would give another proof that the gradient vector is well-defined and uniquely determined. We get that information more generally from Exercise 16.12 without the use of a (local) chart.

Solution of Exercise. (Exercise 16.14, page 16.14) I'm going to start with the intrinsic directional derivatives of the coordinate functions on the circle

$$\mathbb{S}^1 = \{\mathbf{x} = (x_1,x_2)\in\mathbb{R}^2 : x_1^2 + x_2^2 = 1\}.$$

This is a one-dimensional Riemannian manifold where we can obtain an initial covering atlas in a variety of ways. For example, the polar coordinates chart functions

$$\mathbf{p} : (-\pi, \pi) \to \mathbb{S}^1 \qquad \text{by} \qquad \mathbf{p}(t) = (\cos t, \sin t)$$

and

$$\mathbf{q} : (0, 2\pi) \to \mathbb{S}^1 \qquad \text{by} \qquad \mathbf{q}(t) = (\cos t, \sin t)$$

have changes of variables

$$\xi \circ \mathbf{q}\big|_{(0,\pi)\cup(\pi,2\pi)}, \qquad \text{and} \qquad \eta \circ \mathbf{p}\big|_{(-\pi,0)\cup(0,\pi)}$$

both given by the identity. Thus, we can complete $\mathcal{A}_0 = \{((-\pi,\pi), \mathbf{p}), ((0, 2\pi), \mathbf{q})\}$ to a C^k atlas $\mathcal{A}_*^k$ for any $k = 0, 1, 2, \ldots, \infty, \omega$. In such an atlas one also finds stereographic chart functions like

$$\mathbf{p} : \mathbb{R} \to \mathbb{S}^1\backslash\{(0,0,1)\} \qquad \text{by} \qquad \mathbf{p}(x) = \left(\frac{4x}{4+x^2}, \frac{2x^2}{4+x^2} - 1\right),$$

$$\mathbf{p} : \mathbb{R} \to \mathbb{S}^1\backslash\{(0,0,-1)\} \qquad \text{by} \qquad \mathbf{p}(x) = \left(\frac{4x}{4+x^2}, -\frac{2x^2}{4+x^2} + 1\right),$$

and

$$\mathbf{p} : \mathbb{R} \to \mathbb{S}^1\backslash\{(0,0,1)\} \qquad \text{by} \qquad \mathbf{p}(x) = \left(\frac{x}{1+x^2}, \frac{x^2-1}{x^2+1}\right)$$

which are of special interest and the various graph chart functions

$$\mathbf{p} : (-1, 1) \to \mathbb{S}^1 \qquad \text{by} \qquad \mathbf{p}(x) = \left(x, \sqrt{1-x^2}\right),$$

$$\mathbf{p} : (-1, 1) \to \mathbb{S}^1 \qquad \text{by} \qquad \mathbf{p}(x) = \left(x, -\sqrt{1-x^2}\right),$$

$$\mathbf{p} : (-1, 1) \to \mathbb{S}^1 \qquad \text{by} \qquad \mathbf{p}(y) = \left(\sqrt{1-y^2}, y\right),$$

and

$$\mathbf{p} : (-1, 1) \to \mathbb{S}^1 \qquad \text{by} \qquad \mathbf{p}(y) = \left(-\sqrt{1-y^2}, y\right).$$

The unit "circle" $\mathbb{S}^0_P$ at each point $P \in \mathbb{S}^1$ consisists of two elements/filaments/vectors. These may be represented traditionally by

$$\mathbf{u} = \pm P^\perp = \pm(-P_2, P_1)$$

or by $u = \pm[\alpha] \in \mathcal{L}_P\mathbb{S}^1$ with $\alpha(t) = (\cos t, \sin t)$ where $P = (\cos t_0, \sin t_0)$.

If we consider the function $f_1(P) = x^1(P) = P_1$, the directional derivatives associated with the directions $u \in T_P\mathbb{S}^1 \subset \mathbb{R}^2$ are

$$\lim_{t\to t_0} \frac{\cos t - \cos t_0}{t - t_0} = -\sin t_0 = -P_2 \qquad \text{and} \qquad D_{-[\alpha]}f_1(P) = P_2 = f_2(P)$$

where $f_2 : \mathbb{S}^1 \to \mathbb{R}$ by $f_2(P) = x^2(P) = P_2$ is the other coordinate function. Similarly,

$$D_{\pm[\alpha]}f_2(P) = \pm f_1(P).$$

Notice that in this case, f_j can be extended to $\bar{f}_j : \mathbb{R}^2 \to \mathbb{R}$ by the same formula $\bar{f}_j(\mathbf{x}) = x^j(\mathbf{x}) = x_j$ for $j = 1, 2$. Furthermore, the vectors $\mathbf{u} = \pm(-P_2, P_1)$ and $u = \pm[\alpha]$ at $P \in \mathbb{S}^1$ may be considered as elements of $T_P\mathbb{R}^2$ so the values of the (four) directional derivatives given above may be obtained/expressed extrinsically in the ambient space $\mathbb{R}^2$ as

$$D_{\pm[\alpha]}\bar{f}_j(P) = \langle D\bar{f}_j(P), \pm(-P_2, P_1)\rangle_{\mathbb{R}^2} = \langle \mathbf{e}_j, \pm(-P_2, P_1)\rangle_{\mathbb{R}^2} \tag{D.34}$$

for $j = 1, 2$. More generally, the differential $df_j = (df_j)_P =: T_P\mathbb{S}^1 \to \mathbb{R}$ for $j = 1, 2$ also agrees with the traditional differential:

$$\begin{aligned} df_j(v) &= \begin{cases} \|v\|_{T_P\mathbb{S}^1} D_{v/\|v\|} f_j(P), & \|v\|_{T_P\mathbb{S}^1} \neq 0 \\ 0, & v = \mathbf{0} \in T_P\mathbb{S}^1 \end{cases} \\ &= \langle D\bar{f}_j(P), \alpha'(t_0)\rangle_{\mathbb{R}^2} \\ &= d\bar{f}_j(\mathbf{v}) \end{aligned}$$

where $\mathbf{v} = \alpha'(t_0)$ is the traditional vector corresponding to $v = [\alpha]$.

The gradient $Df_j(P)$, that is the intrinsic gradient of f_j with respect to the circle $\mathbb{S}^1$, should not be expected to agree with the traditional gradient $D\bar{f}_j(P)$ of the extension. In fact,

$$D\bar{f}_j(P) = \mathbf{e}_j \in \mathbb{R}^2$$

as appears in (D.34) corresponding to the path $\gamma_j : \mathbb{R} \to \mathbb{R}^2$ by $\gamma_j(t) = P + t\mathbf{e}_j$ for $j = 1, 2$. Note that typically

$$\mathbf{e}_j \in T_P\mathbb{R}^2 \backslash T_P\mathbb{S}^2 \qquad \text{and} \qquad [\gamma_j] \in \mathcal{L}_P\mathbb{R}^2 \backslash \mathcal{L}_P\mathbb{S}^1.$$

On the other hand, $Df_j(P)$ is the element of $\mathcal{L}_P\mathbb{S}^1$ for which

$$df_j(v) = \langle Df_j(P), v\rangle_{T_P\mathbb{S}^1}.$$

As we know $\langle\ \cdot\ ,\ \cdot\ \rangle_{T_P\mathbb{S}^1}$ is the restriction to $T_P\mathbb{S}^1$ of $\langle\ \cdot\ ,\ \cdot\ \rangle_{\mathbb{R}^2}$ either for traditional vectors or for filaments. In any case, we can write

$$\mathbf{e}_j = (\mathbf{e}_j \cdot P)P + (\mathbf{e}_j \cdot P^\perp)P^\perp. \tag{D.35}$$

with $P^\perp \in T_P\mathbb{S}^2$ corresponding to $[\beta] \in \mathcal{L}_P\mathbb{S}^2$ where $\beta(t) = (\cos t, \sin t)$ and $\beta(t_0) = P$. Consequently,

$$\begin{aligned} df_j(v) &= \langle Df_j(P), v\rangle_{T_P\mathbb{S}^1} \\ &= \langle D\bar{f}_j(P), \mathbf{v}\rangle_{T_P\mathbb{R}^2} \\ &= \langle \mathbf{e}_j, \mathbf{v}\rangle_{T_P\mathbb{R}^2} \\ &= \langle (\mathbf{e}_j \cdot P)P + (\mathbf{e}_j \cdot P^\perp)P^\perp, \mathbf{v}\rangle_{T_P\mathbb{R}^2} \\ &= \langle (\mathbf{e}_j \cdot P^\perp)P^\perp, \mathbf{v}\rangle_{T_P\mathbb{R}^2} \\ &= \langle (\mathbf{e}_j \cdot P^\perp)[\beta], v\rangle_{T_P\mathbb{S}^1}. \end{aligned}$$

Thus, we see

$$Df_1(P) = (\mathbf{e}_1 \cdot P^\perp)[\beta] = -P_2[\beta] \qquad \text{and} \qquad Df_2(P) = (\mathbf{e}_2 \cdot P^\perp)[\beta] = P_1[\beta].$$

These filaments correspond to the traditional vectors

$$Df_1(P) = -P_2(-P_2, P_1) \qquad \text{and} \qquad Df_2(P) = P_1(-P_2, P_1).$$

Exercise D.13. Express the orthonormal decomposition (D.35) of traditional vectors in $T_P\mathbb{R}^2$ in terms of filaments in $\mathcal{L}_P\mathbb{R}^2$. Hint: You'll need to introduce a filament corresponding to $P \in T_P\mathbb{R}^2$.

At this point, there are some nice illustrations that can be produced to illustrate the formulas obtained above for the directional derivatives and gradient vectors. First of all, we follow Descartes and append a third spatial direction to the ambient space $\mathbb{R}^2$ with which to represent the values of the functions f_1 and f_2 on the circle $\mathbb{S}^1$ as seen submersed in the x_1, x_2-plane of the resulting ambient $\mathbb{R}^3$. See Figure D.16. Notice the gradient vector $Df_1(\pm\mathbf{e}_1) = \mathbf{0}$ because $f_1(\mathbf{e}_1) = 1$ is a maximum value of f_1 and $f_1(-\mathbf{e}_1) = -1$ is a minimum value of f_1. As $P = (\cos t, \sin t)$ takes values

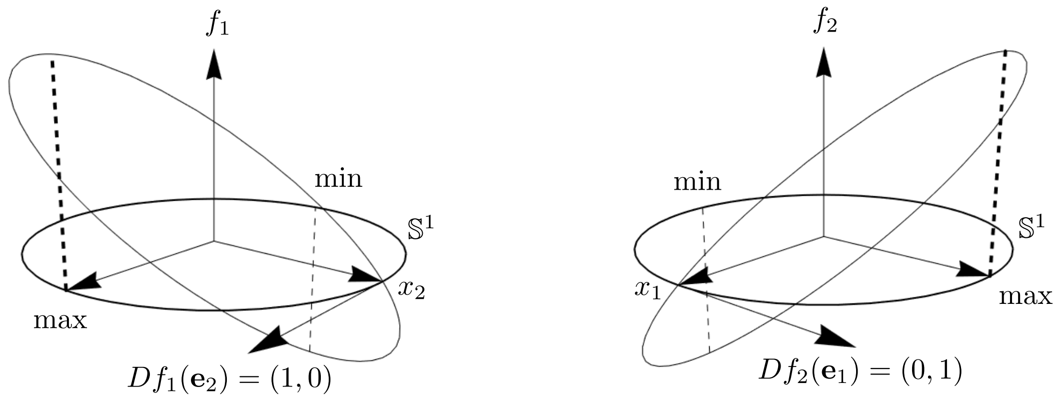

Figure D.16: Plots of the graphs of the coordinate functions over $\mathbb{S}^1 \subset \mathbb{R}^2$.

starting at $\mathbf{e}_1$ with t increasing from $t = 0$, the magnitude of $Df_1(P)$ increases with P_1 and points in the clockwise direction until $Df_1(\mathbf{e}_2)$ takes the value $-\alpha'(\pi/2) = \mathbf{e}_1$ when $P = \alpha(\pi/2) = \mathbf{e}_2$ as indicated on the left in Figure D.16.

Second, it will be noticed that the gradient field Df_j is the projection of the gradient field $D\bar{f}_j = \mathbf{e}_j$ onto $T_P\mathbb{S}^1$ for $j = 1, 2$. This is illustrated in Figure D.17. Notice that when $Df_j(P) \neq \mathbf{0}$, then $u = Df_j(P)/\|Df_j(P)\|_{T_P\mathbb{S}^1} \in$

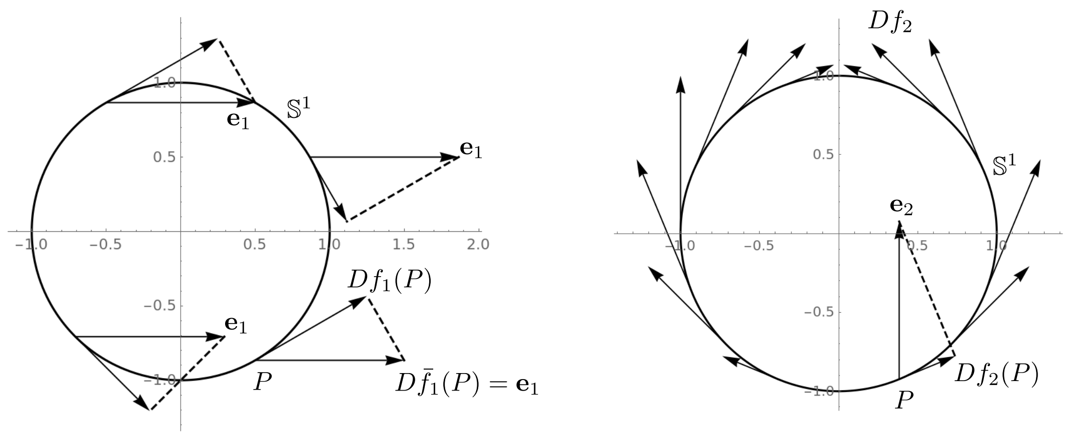

Figure D.17: Plots of the gradient vectores of the coordinate functions on $\mathbb{S}^1 \subset \mathbb{R}^2$.

$\mathbb{S}^0_P$ picks the direction (from among the two possible choices in $\mathbb{S}^0_P$) of maximum increase of f_j and $\|Df_j(P)\|$ is the value of the directional derivative $D_u f_j(P)$.

It may be noticed that we have not used any of the chart functions mentioned at the beginning of this solution at all. We have in fact derived a formula for the intrinsic gradient in terms of a chart function in the solution of Exercise 16.13 above, and I for one am keen to use that formula. Starting with a point $P = (\cos t_0, \sin t_0) = \mathbf{p}(t_0)$ with $-\pi < t_0 < \pi$, the metric coefficient is $g \equiv 1$ because in this case the metric tensor is inherited from $\mathbb{R}^2$ and if we take path $\alpha : I \to \mathbb{S}^1$ with

$$\begin{aligned} &\alpha(t_1) = P, \\ &\mathbf{v} = \alpha'(t_1), \quad \text{and} \\ &v = [\alpha], \end{aligned}$$

and $\beta : J \to \mathbb{S}^1$ with

$$\begin{aligned} &\beta(t_2) = P, \\ &\mathbf{w} = \beta'(t_2), \quad \text{and} \\ &w = [\beta], \end{aligned}$$

then

$$\mu_P([\alpha], [\beta]) = \langle g(\xi \circ \alpha)'(t_1), (\xi \circ \beta)'(t_2) \rangle_{\mathbb{R}} = \langle \alpha'(t_1), \beta'(t_2) \rangle_{\mathbb{R}^2}. \tag{D.36}$$

The coordinate function $\xi = \mathbf{p}^{-1} : \mathbb{S}^1 \backslash \{(-1, 0)\} \to \mathbb{R}$ is not so simple in this case, but for $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 > 0$, we can write

$$\xi \circ \alpha(t) = \tan^{-1} \frac{\alpha_2(t)}{\alpha_1(t)}$$

and

$$\begin{aligned} (\xi \circ \alpha)' &= \frac{1}{1 + \frac{\alpha_2^2}{\alpha_1^2}} \left(\frac{\alpha_1 \alpha_2' - \alpha_2 \alpha_1'}{\alpha_1^2} \right) \\ &= \frac{\alpha_1 \alpha_2' - \alpha_2 \alpha_1'}{\alpha_1^2 + \alpha_2^2} \\ &= \alpha_1 \alpha_2' - \alpha_2 \alpha_1', \end{aligned}$$

so that

$$(\xi \circ \alpha)'(t_1) = P_1 \alpha_2'(t_1) - P_2 \alpha_1'(t_1). \tag{D.37}$$

Similarly,

$$(\xi \circ \beta)'(t_2) = P_1\beta_2'(t_2) - P_2\beta_1'(t_2). \tag{D.38}$$

Exercise D.14. Show the identities (D.37) and (D.38) hold for the polar coordinates chart function $\mathbf{p}$ even when $P_1 = \alpha_1(t_1) = \beta_1(t_2) \leq 0$.

On the other hand, it is also true that $|\alpha|^2 = 1$ so that $\langle \alpha, \alpha' \rangle_{\mathbb{R}^2} = 0$. Thus,

$$\alpha' = \langle \alpha', \alpha^\perp \rangle_{\mathbb{R}^2} \alpha^\perp \qquad \text{and} \qquad \alpha'(t_1) = [-P_2\alpha_1'(t_1) + P_1\alpha_2'(t_1)](-P_2, P_1).$$

Similarly,

$$\beta'(t_2) = [-P_2\beta_1'(t_2) + P_1\beta_2'(t_2)](-P_2, P_1).$$

Therefore,

$$\langle g(\xi \circ \alpha)'(t_1), (\xi \circ \beta)'(t_2) \rangle_{\mathbb{R}} = g[P_1\alpha_2'(t_1) - P_2\alpha_1'(t_1)][P_1\beta_2'(t_2) - P_2\beta_1'(t_2)],$$

and on the other hand

$$\langle \alpha'(t_1), \beta'(t_2) \rangle_{\mathbb{R}^2} = [-P_2\alpha_1'(t_1) + P_1\alpha_2'(t_1)][-P_2\beta_1'(t_2) + P_1\beta_2'(t_2)].$$

In view of (D.36) this means $g = 1$. Referring then to the coefficient expression (D.32) when $n = 1$ and $g^{-1} = 1$, we note that $f_j \circ \mathbf{p}(t) = f_j(\cos t, \sin t)$ and

$$D(f_j \circ \mathbf{p})(t) = \mathbf{e}_j \cdot (-\sin t, \cos t) = \langle \mathbf{e}_j, \alpha'(t) \rangle_{\mathbb{R}^2}$$

where $\alpha(t) = (\cos t, \sin t)$. In particular, at $P = (\cos t_0, \sin t_0)$ we obtain the coefficient

$$c = D(f_j \circ \mathbf{p})(t_0)\ g^{-1} = \mathbf{e}_j \cdot (-\sin t_0, \cos t_0) = \langle \mathbf{e}_j, P^\perp \rangle_{\mathbb{R}^2}.$$

Finally, the coordinate induced basis for $T_P\mathbb{S}^1$ at $P = (\cos t_0, \sin t_0)$ has traditional form $\mathbf{v} = (-\sin t_0, \cos t_0) = \alpha'(t_0) = P^\perp$ corresponding to $v = [\alpha]$. Thus, we have obtained complete agreement with our previous calculation(s) of the gradient(s) of $f_j : \mathbb{S}^1 \to \mathbb{R}$ for $j = 1, 2$:

$$Df_j(P) = cv = \langle \mathbf{e}_j, P^\perp \rangle_{\mathbb{R}^2}\ [\alpha]$$

and

$$c\mathbf{v} = \langle \mathbf{e}_j, P^\perp \rangle_{\mathbb{R}^2}\ P^\perp.$$

Exercise D.15. Repeat the calculation above using stereographic chart functions and/or graph chart functions for points in $\mathbb{S}^1$.