Appendix C

Thoughts on $\mathcal{B}$

In casting about for ways to understand and/or illustrate the Riemannian manifold $\mathcal{B}$ introduced in Chapter 3 certain very nice suggestions have been made, and I am going to record, expand on, and comment on those suggestions here.

C.1 Travis' idea

Based on certain calcuations of $\text{length}_{\mathcal{B}}[\alpha]$ given by

$$\underset{\mathcal{B}}{\text{length}}[\alpha] = \int_{(a,b)} \frac{4}{4+|\alpha|^2}\ |\alpha'| \tag{C.1}$$

and the fact that

$$\int_0^\infty \frac{4}{4+t^2}\, dt = \pi$$

in particular, Travis Driver suggested[1] $\mathcal{B}$ has something to do with a **hemisphere**. This turns out to be correct. It was my initial inclination that more could be internalized concerning the basic concept of a Riemannian manifold without making such a connection,[2] and hopefully there is still plenty of opportunity for that. On the other hand, there is absolutely no reason not to pursue this line of inquiry. I would suggest the first thing to do is write

[1]Technically, I believe Travis suggested something to the effect that $\mathcal{B}$ "models" a hemisphere, but I'm simplifying here.

[2]Specifically along the lines of Ty's work in the previous appendix.

down what one means by a hemisphere. Here is a possibility:

$$M = \mathbb{S}_r^{2,+} = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1,\ x_3 > 0\}.$$

There is an open hemisphere. Second, if one is to make a comparison between M and $\mathcal{B}$, then one should seek a **bijection** $\Psi : \mathcal{B} \to M$ or a bijection $\mathbf{q} : B_1(\mathbf{0}) \to M$ which roughly speaking amounts to the same thing.

I won't say much more, because I didn't get much more out of Travis. I will say this is a reasonable idea. In particular,

1. M is a Riemannian manifold (whatever that is).
2. It is true that there is a notion of lengths $\text{length}_M[\alpha]$ of paths $\alpha \in C^1([a, b] \to M)$, and
3. There is a reasonable notion of what C^1 means in this case.

Furthermore, if one could verify that paths in $\mathcal{B}$ corresponding under such a bijection to paths in M have the same lengths, that would be exceedingly suggestive.

I will perhaps offer one last cautionary comment. One should keep in mind that even if such an exceedingly suggestive bijection is obtained, then that does not quite mean $\mathcal{B}$ and M are the "same" Riemannian manifold. It may turn out that $\mathcal{B}$ and M are "isomorphic" and/or "isometric" Riemannian manifolds (terms we have yet to define), but in no case will M and $\mathcal{B}$ be any more "the same" Riemannian manifold than are $\mathcal{B}$ and $\mathcal{C}$.

C.2 Ruijia's idea

If I understood Ruijia Cao correctly, he suggested a specific form for the bijection $\Psi : \mathcal{B} \to M$, and it was essentially this:

$$\Psi(\mathbf{x}) = \left(rx_1, rx_2, r\ \sqrt{1 - |\mathbf{x}|^2}\right)$$

where $\mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0})$, which as we know is the "same" as $\mathcal{B}$ as a point set, though different when it comes to calculating lengths. Concerning this suggestion I point out the following: The path in M corresponding via this bijection to the radial path parameterized by $\alpha(t) = t(1, 0) \in \mathcal{B}$ for $0 \le t \le a < 1$ is

$$\beta(t) = \left(rt, 0, r\sqrt{1 - t^2}\right).$$

For this path, the Euclidean length in $M = \mathbb{S}_r^{2,+}$ is given by

$$\begin{aligned}\text{length}_M[\beta] &= \int_0^a |\beta'(t)|\, dt \\ &= r \int_0^a \frac{1}{\sqrt{1-t^2}}\, dt \\ &= r \sin^{-1}(a)\end{aligned}$$

since

$$\beta'(t) = \left(r, 0, -r\frac{t}{\sqrt{1-t^2}} \right).$$

This calcuation can be "seen" quite easily in an illustration of M. But if we wish to have this match the Riemannian length in $\mathcal{B}$, then we need (to choose r so that)

$$\text{length}_M[\beta] = r\sin^{-1}(a) = \text{length}_{\mathcal{B}}[\alpha] = 2\tan^{-1}\left(\frac{a}{2}\right).$$

This is impossible. I will include some further comments in a section below.

C.3 Matthew's idea and example $\mathcal{D}$

Matthew Sumanen had a quite different idea. Rather than consider $\mathcal{B}$ directly he suggests casting about for some compelling relation between $\mathcal{B}$ and a different Riemannian manifold which he refers to as the **Poincaré disk**. This is also a very interesting idea. In fact, one "model" for the Poincaré disk is the following: Consider $\mathcal{D}$ which as a point set is $B_1(\mathbf{0}) \subset \mathbb{R}^2$. Make a matrix assignment

$$(g_{ij}) = \frac{16}{(4-|\mathbf{x}|^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that

$$\begin{aligned}\text{length}_{\mathcal{D}}[\alpha] &= \int_{(a,b)} \sqrt{\langle \alpha', \alpha' \rangle_{\mathcal{D}}} \\ &= \int_{(a,b)} \sqrt{\langle (g_{ij})\ \alpha',\ \alpha' \rangle_{\mathbb{R}^2}} \\ &= \int_{(a,b)} \frac{4}{4-|\alpha|^2}\ |\alpha'|.\end{aligned}$$

The (much more than) superficial resemblance of this formula to (C.1) strongly suggests Matthew is onto something. What that is remains to be seen.

I will offer, however, the calculation of the Riemannian length[3] of radial segments in $\mathcal{D}$. Let $\alpha(t) = t(\cos\theta, \sin\theta)$ for $0 \le t \le a$. Then

$$\begin{aligned}\text{length}_{\mathcal{D}}[\alpha] &= \int_0^a \frac{4}{4-t^2}\,dt \\ &= \int_0^a \frac{1}{1-(t/2)^2}\,dt \\ &= 2\int_0^{a/2} \frac{1}{1-u^2}\,du \\ &= 2\tanh^{-1}\left(\frac{a}{2}\right).\end{aligned}$$

It will be observe that rather than having a finite bound for all positive a, the choice of the radius $r = 2$ I have chosen in $B_2(\mathbf{0})$ is fundamentally inherent in that

$$\lim_{a\nearrow 2} \text{length}_{\mathcal{D}}[\alpha] = +\infty.$$

With this, I look forward to hearing what else Matthew comes up with in his comparison. There are also some more related comments below.

C.4 Tether-and-circle construction

Hemisphere and certain smaller pieces of spheres given as a graph over a disk, as suggested by Travis and considered by Ruijia, have a kind of interesting decomposition property. Specifically, let us set

$$h(\mathbf{x}) = r - \sqrt{r^2 - |\mathbf{x}|^2},$$

for $\mathbf{x} = (x_1, x_2) \in B_r(\mathbf{0})$ for some $r > 0$. The graph of h is then the hemospherical surface $M = \mathbb{S}_r^{2,+}$ considered above. There is a **tether function** $f : [0, r] \to \mathbb{R}$ by

$$f(x) = r - \sqrt{r^2 - x^2},$$

[3]The value of $\text{length}_{\mathcal{D}}[\alpha]$ is also known as the **hyperbolic length** in this case because the Poincaré disk is also known as the hyperbolic disk, though technically manifolds of this sort are usually considered with a slightly different matrix assignment on $B_1(\mathbf{0})$. There is also a version on a half-plane called the hyperbolic plane.

and passing through each point $(t, 0, f(t))$ for $0 < t < r$ on the curve Γ parameterized by

$$\alpha(t) = (t, 0, f(t))$$

for $0 \le t < r$, which happens to lie on $M = \mathbb{S}_r^{2,+}$, there is a unique circle of radius t in a horizontal plane and in M. This suggests a generalization.

Consider a surface constructed by passing through each point $(t, 0, f(t))$ determined by any even function $f \in C^2[-a, a]$ a circle $\{(x_1, x_2, f(t)) : x_1^2 + x_2^2 = |t|\}$. This surface is a surface of rotation and is parameterized by

$$X(t, \theta) = (t\cos\theta, t\sin\theta, f(t)).$$

Naturally, the Euclidean length of one of the "level circles" or **longitudinal circles** is $2\pi t$. The length (Riemannian length that is) of the "radial" path along the surface from $\mathbf{0} = (0, 0, 0) \in \mathbb{R}^3$ to the circle is given by

$$\int_{(0,t)} \sqrt{1 + f'^2}.$$

As we have seen, this procedure, at least where f determines a circular arc and the surface is a spherical cap, does not immediately give a bijection with the Riemannian manifold $\mathcal{B}$.

I'm going to suggest, like Matthew however, something a little different and somewhat unexpected.

Exercise C.1. Each "tether-and-circle" surface determines a matrix assignment on a disk which can be used to calculate Riemannian lengths on the disk which will be in a one-to-one bijective correspondence with the lengths of curves on the surface.

(a) Take an appropriate disk and find the matrix assignment corresponding to a given "tether-and-circle" surface.

(b) Completion of part **(a)** above should suggest a certain family of axially symmetric matrix assignments leading to Riemannian manifolds determined on disks (like $\mathcal{B}$ and $\mathcal{D}$) which can indeed (easily) be put into one-to-one correspondence with axially symmetric surfaces. Can you characterize these matrix assignments?

As something of an aside, and returning to Travis' original idea, the "reverse" or "converse" question of the exercise above involving taking a

Riemannian manifold $\mathcal{M}$ determined by a matrix assignment on a region in Euclidean space $\mathbb{R}^2$ (like a ball) and finding a bijection $\Psi : \mathcal{M} \to M$ where M is a surface in $\mathbb{R}^3$ is an example of what is called the **local embedding problem**, and this problem has historically played a central role in the development of Riemannian geometry.

The last thing I will point out is that for certain matrix assignments like the one Matthew suggests giving the hyperbolic disk the tether-and-circle construction can be seen to be impossible in an interesting way. Let's go back to $\mathcal{B}$ for a moment, and then we'll consider the hyperbolic disk $\mathcal{D}$.

We weren't quite able to make the tether-and-circle construction work for $\mathcal{B}$, but let us consider some aspect of it once again. Given a point $(a, 0) \in \mathcal{B}$, we have computed a Riemannian radius

$$\text{radius}_{\mathcal{B}}(a) = 2\tan^{-1}\left(\frac{a}{2}\right).$$

We can (and one of the exercises asks you to) calculate a Riemannian circumference also:

$$\text{circumference}_{\mathcal{B}}(a) = \int_0^{2\pi} \frac{4a}{4+a^2}\,dt = \frac{8\pi a}{4+a^2}.$$

Now, if we imagine the points in $\mathcal{B}$ correspond to points in an axially symmetric surface constructed by a tether-and-circle construction, then the circle

$$\{\mathbf{x} \in \mathcal{B} : |\mathbf{x}| = a\} \subset \mathcal{B}$$

should correspond to a Euclidean circle

$$\{(x_1, x_2, f(t)) : x_1^2 + x_2^2 = t^2\} \subset M$$

in the surface. If the radius of this circle is to match $\text{circumference}_{\mathcal{B}}(a)$ we must have

$$t = \frac{4a}{4+a^2}.$$

On the other hand, if the tether length

$$\int_{(0,t)} \sqrt{1+f'^2}$$

is to match the Riemannian radius $\text{radius}_{\mathcal{B}}(a)$, then we must have

$$\int_{(0,t)} \sqrt{1+f'^2} = 2\tan^{-1}\left(\frac{a}{2}\right).$$

Notice that

$$\int_{(0,t)} \sqrt{1+f'^2} \geq t = \frac{4a}{4+a^2}.$$

This suggests a comparison of the quantities

$$\mathrm{radius}_{\mathcal{B}}(a) = 2\tan^{-1}\left(\frac{a}{2}\right) \qquad \text{and} \qquad \mathrm{radius}_{\mathbb{R}^3}(a) = \frac{4a}{4+a^2}.$$

The plot on the left in Figure C.1 shows

$$\mathrm{radius}_{\mathbb{R}^3}(a) < \mathrm{radius}_{\mathcal{B}}(a)$$

for $0 < a < 1$. This means a tether-and-circle construction is possible in principle. The tether for the horizontal circle of radius $\mathrm{radius}_{\mathbb{R}^3}(a)$ in $\mathbb{R}^3$

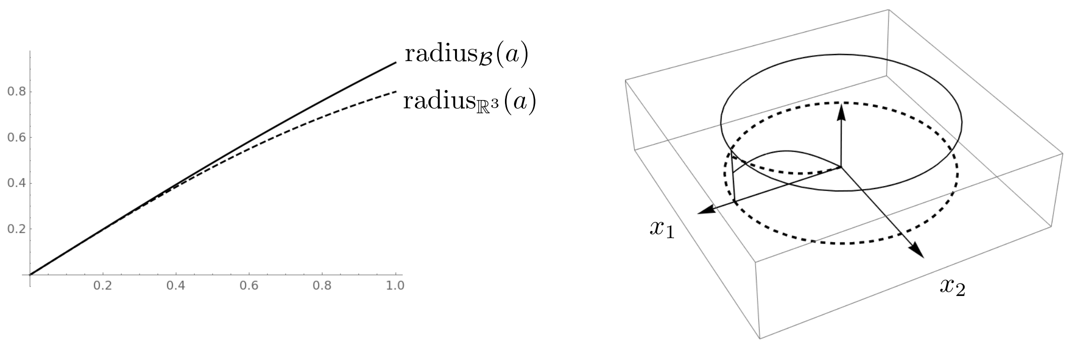

Figure C.1: Comparison of $\mathrm{radius}_{\mathcal{B}}(a)$ and $\mathrm{radius}_{\mathbb{R}^3}(a)$ (left). In principle, the circle corresponding to the circle in $\mathcal{B}$ of Euclidean radius a can be at any height between a maximum height $x_3 = M$ and $x_3 = -M$. The dashed circle shows the circle at height $x_3 = 0$ while the solid circle is at the maximum height for $a = 0.9$. The solid meridian and the dashed meridian are curves each of which has length the correct intrinsic radius $\mathrm{radius}_{\mathcal{B}}(a) = \mathrm{length}_{\mathcal{B}}[\alpha]$ where α is the radial segment.

should be of length $\mathrm{radius}_{\mathcal{B}}(a)$. This means such a horizontal circle can have maximum height

$$M = \sqrt{[\mathrm{radius}_{\mathcal{B}}(a)]^2 - [\mathrm{radius}_{\mathbb{R}^3}(a)]^2} > 0.$$

A circle at this height is joined to the origin by only tether curve of length $\mathrm{radius}_{\mathcal{B}}(a)$ which is a straight line. In particular chosing this height or the height $x_3 = -M$ for the circle corresponding to any given $a > 0$ determines the tether function f uniquely and forces the tether-and-circle surface to be conical, in fact in this case, the surface must be a cone over the circle. In this case, however, the corresponding linear function f cannot extend to have an even extension in $C^1[-\text{radius}_{\mathbb{R}^3}(a), \text{radius}_{\mathbb{R}^3}(a)]$. For heights x_3 satisfying $-M < x_2 < M$ multiple choices for f are in principle possible. Any curve $\{(x, 0, f(x)) : 0 \le x \le \text{radius}_{\mathbb{R}^3}(a)\}$ with

$$\int_0^{\text{radius}_{\mathbb{R}^3}(a)} \sqrt{1 + [f'(x)]^2}\, dx = \text{radius}_{\mathcal{B}}(a)$$

may be considered a candidate, though the regularity at $x = 0$ must still be taken into account.

If one carries out this procedure with the Euclidean matrix assignment in $B_1(\mathbf{0})$, then one has imposed the Euclidean relationship

$$\text{radius}_{\mathbb{R}^3}(a) = a = \text{radius}_{B_1(\mathbf{0})}(a).$$

There is only one choice for the circle and the tether for each $a > 0$, namely each circle must be located at height $x_3 = 0$ giving back a copy of the disk $B_1(\mathbf{0})$ in the (x_1, x_2) plane.

If one attempts this same construction for Matthew's example (or more properly my example $\mathcal{D}$) something interesting happens. As indicated on the left in Figure C.2 we find

$$\text{radius}_{\mathbb{R}^3}(a) = \frac{4a}{4 - a^2} > 2\tanh^{-1}\left(\frac{a}{2}\right) = \text{radius}_{\mathcal{D}}(a).$$

In this way, one can rule out any kind of tether-and-circle construction. One may perhaps consider a non-circular curve of length $\text{radius}_{\mathcal{D}}(a)$ lying in a cylinder of smaller radius. For example, a curve like

$$\Gamma = \left\{(x_1, x_2, x_1^2 - x_2^2) \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} = \text{radius}_{\mathbb{R}^3}(a)\right\}$$

for some

$$\text{radius}_{\mathbb{R}^3}(a) < \frac{4a}{4 - a^2}.$$

In this case, however, one cannot easily determine a family of tether curves with constant length $\text{radius}_{\mathcal{D}}(a)$ connecting the origin to such a curve Γ. If one is to obtain identification with an embedded surface in $\mathbb{R}^3$ even locally, something more complicated must be done. Something is essentially different about these Riemann surfaces.

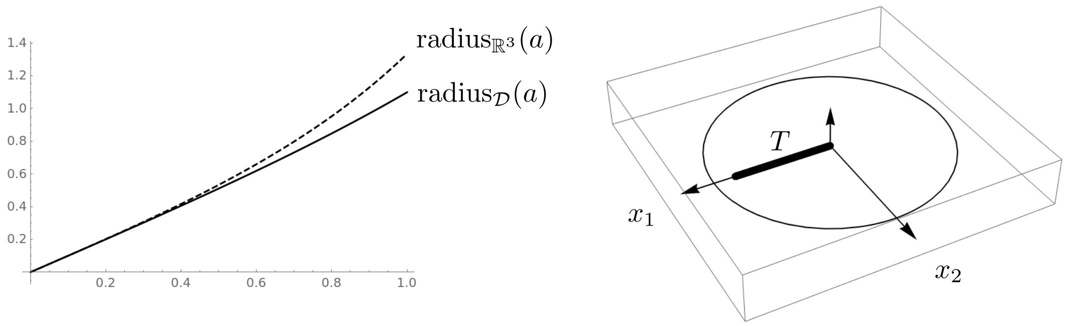

Figure C.2: Comparison of radius$_{\mathcal{D}}(a)$ and radius$_{\mathbb{R}^3}(a)$ (left). Here it is impossible for a tether to reach a circle in a horizontal plane of the appropriate radius radius$_{\mathbb{R}^3}(a)$. The heavy tether segment T of length radius$_{\mathcal{D}}(a)$ extends from the origin and is straining to reach the circle but is too short.

Exercise C.2. Show the saddle shaped surface

$$\left\{(x_1, x_2, x_1^2 - x_2^2) : (x_1, x_2) \in \mathbb{R}^2\right\}$$

can be used to locally induce a matrix assignment on an open disk $B_\epsilon(\mathbf{0})$ giving a one-to-one correspondence of lengths of paths on the surface with the Riemannian lengths of the corresponding paths in the disk calculated using the induced matrix assignment. Show, however, that this matrix assignment is not axially symmetric in the disk.