The Equation of Constant Mean Curvature  
(Graphs Part II)

Math 6456 Differential Geometry  
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We wish now to obtain apriori estimates for solutions \( u \) of the quasilinear Dirichlet problem

\[
(\ast) \quad \begin{cases} 
Q[u] := \sum A_{ij}(Du) D_i D_j u = f & \text{on } \Omega \\
\left. u \right|_{\partial \Omega} = \phi 
\end{cases}
\]

where it is assumed that for a given \( u \in C^1(\bar{\Omega}) \), the coefficient matrix \( (A_{ij}(Du)) \) depends smoothly on \( Du \) and is uniformly elliptic with ellipticity constant \( \epsilon_0 = \epsilon_0(Du) \) satisfying

\[
\frac{1}{\epsilon_0(Du)} \leq C_0(\|Du\|_{C^0(\Omega)})
\]

(and \( C_0 \) is a continuous increasing function on \([0, \infty)\)).

We still assume \( \Omega \) is a bounded smooth domain (at least \( C^2 \)) and that \( f \in C^2(\Omega) \).

**Exercise 1** Show that if \( u \in C^2(\bar{\Omega}) \) and \( \Omega_0 \subset \subset \Omega \), then \( u \in C^\infty(\Omega_0) \).

In view of this exercise, we find ourselves in a curious situation: *If we have a solution, it is smooth, but we cannot say that we have a solution until we get the relatively low order estimate required by condition \((A_0)\) of Theorem 1 (the reduction theorem) stated previously; namely, we need an estimate on \( \|u\|_{C^{1,\beta}(\Omega)} \) for some \( \beta \). Recall that by definition

\[
\|u\|_{C^{1,\beta}(\Omega)} = \|u\|_{C^0(\Omega)} + \sup_{j=1,\ldots,n} \|D_j u\|_{C^0(\Omega)} + \sup_{j=1,\ldots,n} [D_j u]_{\beta}.
\]

We described above how to get an estimate on the \( C^0 \) norm of \( u \) by using the maximum principle when \( f \) was constant.

**Exercise 2** Show that in the case of more general \( f \) one has

\[
\|u\|_{C^0(\Omega)} \leq \sup_{\partial \Omega} |u| + e^{\text{diam}(\Omega)} \sup_{\Omega} \left( \frac{|f|}{\epsilon_0} \right)
\]

where \( \epsilon_0 = 1/(1 + |Du|^2)^{3/2} \).
This kind of estimate reduces the height estimate to that of $\|D_j u\|_{C^0(\Omega)}$. That was good enough for the reduction of existence (Theorem 1) to the Leray-Schauder fixed point theorem, which should not be too surprising because after all $(A_0)$ already assumed a bound on $\|Du\|$. This kind of estimate, however, will be inadequate for the general verification of $(A_0)$. Here is a much stronger result for which we discuss two proofs:

**Theorem 2 (J. Serrin)** If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of the Dirichlet problem

\[
(DP) \quad \begin{cases}
Q[u] := \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 2H & \text{on } \Omega \\
u|_{\partial \Omega} = \phi
\end{cases}
\]

with $H$ a nonzero constant, then

\[
u \leq \max_{\partial \Omega} \phi + \frac{1}{|H|} \quad \text{on } \Omega
\]

with equality only if $\inf \partial \Omega$ is a circle, $\phi$ is constant, and the graph of $u$ is a hemispherical cap.

The main tool we wish to emphasize in the proofs of this result is the maximum principle (which is the same thing we used to get the other height estimate). A version which is specially suited to quasilinear equations will be useful.

**Lemma 2 (the comparison principle)** If

\[
\begin{cases}
Q[u] \geq Q[v] & \text{on } \Omega \\
u|_{\partial \Omega} \leq v|_{\partial \Omega}
\end{cases}
\]

then $u \leq v$ on $\Omega$ with equality only if $u \equiv v$.

**Exercise 3** Show that if $u$ is a solution of (DP) with $H = 0$, then $\min_{\partial \Omega} \phi \leq u(x) \leq \max_{\partial \Omega} \phi$ for $x \in \Omega$ with equality only if $u \equiv \text{constant}$.

**Exercise 4** Use the comparison principle to show that solutions of (DP) are unique (assuming existence).

The first proof of Theorem 2, which is Serrin’s original proof, uses a little more about the geometry of mean curvature; these are things which are really useful to know in general.
More on Mean Curvature; The Gauss Map

We have seen that given a piece of surface $S$ expressed as the graph of a function $u$, the sum of the normal curvatures of any two planar curves in the surface crossing orthogonally at the point expressed by $p = (x, u(x))$ is given by

$$\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right). \quad (2)$$

(To get the normal curvature of a curve on a surface, find some function $\gamma : (-\epsilon, \epsilon) \to S$ which parameterizes the curve by arclength; then $\dot{\gamma}$ is the curvature vector and the normal curvature is defined by be $\kappa_n = \dot{\gamma} \cdot N$. The fact that this number is the same for all curves on $S$ passing through $p$ and tangent to each other is called Meusnier’s Theorem (pronounced “Moon-yea”). Note how we got around this in the previous discussion by taking the curves to be slices of the surface by planes containing the normal $N$.)

**Exercise 5** Prove Meusnier’s theorem. Hint: The unit normal is defined all along the curve, so we can think of it as a function of arclength $s$. Compute $\frac{d}{ds}(\dot{\gamma} \cdot N)$.

We now consider the expression for $2H$ given in (2) from a more abstract (and slightly algebraic) point of view. Our discussion begins with a general parameterization $X : U \to S \subset \mathbb{R}^3$ and the associated Gauss map $N : U \to S^2$, or normal to $S$, where $S^2$ is the unit sphere in $\mathbb{R}^3$. If we assume that $U$ is a subset of the $u,v$-plane, then

$$N = \frac{X_u \times X_v}{|X_u \times X_v|}.$$  

The differential of the Gauss map is defined to be the linear map $dN_p : T_pS \to T_pS$ given by

$$dN_p(w) = N'$$

where $T_pS$ is the tangent plane to $S$ at $p$, the vector $w \in T_pS$, and $N'$ denotes the vector in $T_pS$ obtained as follows:

Let $\gamma : (-\epsilon, \epsilon) \to U$ parameterize a curve in $U$ for which $X \circ \gamma(0) = p$ and

$$X' := \left. \frac{d}{dt} (X \circ \gamma) \right|_{t=0} = w. \quad (3)$$

$$N' := \left. \frac{d}{dt} (N \circ \gamma) \right|_{t=0}. \quad (4)$$

**Exercise 6** 1. Show that given $w \in T_pS$, there exists a curve $\gamma$ satisfying (3).
2. Show that the value of $N'$ given in (4) does not depend on the choice of $\gamma$ as long as $\gamma$ satisfies (3).

3. Show that $N'$, as defined in (4), is in $T_p S$.

4. Show that the mapping $dN_p : T_p S \to T_p S$ is linear.

**Exercise 7** Show that $dN_p$ is symmetric in the sense that $dN_p(v) \cdot w = v \cdot dN_p(w)$ for every pair of vectors $v, w \in T_p S$. Hint: \{X_u, X_v\} is a basis for $T_p S$ and $dN_p(X_u) = N_u$.

**Exercise 8** Given a symmetric linear map $L : V \to V$ of a two dimensional inner product space $V$ with inner product $\langle \cdot, \cdot \rangle$, there is an orthonormal basis of eigenvectors $w_1, w_2 \in V$ such that

$$\langle Lw_1, w_1 \rangle = \max_{w \in V; |w|=1} \langle Lw, w \rangle$$

and

$$\langle Lw_2, w_2 \rangle = \min_{w \in V; |w|=1} \langle Lw, w \rangle.$$

Hint: Write $w = \cos \theta e_1 + \sin \theta e_2$.

Now, if $\gamma$ is an arclength parameterization of any curve with $\gamma(0) = p$, so that the curvature vector of this curve at $p$ is $\vec{\gamma}(0)$, and the signed normal curvature of $\gamma$ with respect to $N$ in the plane spanned by $\vec{\gamma}$ and $N$ (n.b., Meusnier’s Theorem) is

$$\kappa = \vec{\gamma} \cdot N,$$

then

$$\kappa = \vec{\gamma} \cdot N = -\dot{\gamma} \cdot \dot{N} = -\dot{\gamma} \cdot dN_p(\dot{\gamma}).$$

Thus, we arrive at an expression for the mean curvature using the notion of the trace of a linear transformation familiar from linear algebra:

$$2H = -\text{tr} dN_p.$$ 

We also obtain a useful algebraic expression for the Gauss curvature:

**Exercise 9** Show that the maximum and minimum values $\kappa_1$ and $\kappa_2$ of the normal curvatures at $p$ satisfy

$$K = \kappa_1 \kappa_2 = \det dN_p.$$

Hint: Use Exercise 8.

**Definition 1** The maximum and minimum normal curvatures are called principal curvatures and the corresponding eigenvectors are called principal directions.

A point at which the maximum and minimum normal curvatures (and hence all normal curvatures) are equal ($\kappa_n \equiv \kappa_1 = \kappa_2$) is called an umbilic point.
The following identities are now easily obtained.

**Exercise 10** Show that for any local parameterization $X : U \rightarrow S$,

1. $X_u \times N_v - X_v \times N_u = X_u \times dN_p(X_v) - X_v \times dN_p(X_u) = -2H(X_u \times X_v)$.
2. $N_u \times N_v = dN_p(X_u) \times dN_p(X_v) = K(X_u \times X_v)$.

We are now in a position to verify one of Serrin’s main tools:

**Lemma 3 (Bonnet’s parallel surface construction)** If $(x_0, u(x_0))$ is not an umbilic point on the graph of $u$, then there is some $r > 0$ such that the mapping $\tilde{X}(x) = (x, u(x)) + \frac{1}{H}N$ regularly parametrizes a constant mean curvature graph on $B_r(x_0)$ with mean curvature $-H$.

**Exercise 11** Prove Bonnet’s Lemma as follows.

1. First let $X : U \rightarrow \mathbb{R}^3$ parameterize any surface and set $\tilde{X} = X + cN$ where $c$ is a constant. Show that $\tilde{X}_u \times \tilde{X}_v = \lambda(X_u \times X_v)$ where $\lambda = 1 - 2Hc + Kc^2$.
2. Assume that $\lambda$ in the construction above is nonzero on $U$. Then $\tilde{X}$ is a regular parameterization whose image has mean curvature $\tilde{H}$ satisfying $\tilde{H}(\tilde{X}_u \times \tilde{X}_v) = \text{sign}(\lambda)(H - cK)(X_u \times X_v)$.
3. Take $c = 1/H$ and show that $\lambda > 0$ except at umbilic points, and verify Bonnet’s conclusion.

Note: It is instructive to consider Bonnet’s construction for a cylinder.

We are now ready to tackle Serrin’s beautiful argument.

**Serrin’s Proof of Theorem 2:** If $H > 0$, then the comparison principle applies with $v \equiv \max_{\partial \Omega} \phi$, and clearly equality in (1) never occurs. We may thus restrict to the case $H < 0$.

If we can show that the parallel surface parameterized on the open set $\Omega$ lies entirely below the plane $z = \max_{\partial \Omega} u$, then we would have the strict estimate. Of course this is not true in the case of a hemispherical cap, but let us proceed in this direction and see if this exceptional situation appears in the argument.
Let us assume, on the contrary, that for some point $x \in \Omega$ we have

$$u(x) + \frac{1}{H \sqrt{1 + |Du(x)|^2}} \geq \max_{\partial \Omega} u.$$  

One possibility is that we have equality here and $x$ is a point at which the parallel surface has maximal third component. If this is not the case, then there is some (other) $x \in \Omega$ and some $\epsilon > 0$ for which

$$u(x) + \frac{1}{H \sqrt{1 + |Du(x)|^2}} \geq \max_{\partial \Omega} u + \epsilon.$$  

We claim that such points are isolated from the boundary of $\Omega$. In fact, since $H < 0$ and $u$ is continuous up to the boundary,

$$\lim_{x \to \partial \Omega} \left[ u(x) + \frac{1}{H \sqrt{1 + |Du(x)|^2}} \right] \leq \max_{\partial \Omega} u.$$  

We conclude that we can find a (maximal) point $x_0 \in \Omega$ such that

$$u(x_0) + \frac{1}{H \sqrt{1 + |Du(x_0)|^2}} = \max_{\Omega} \left[ u(x) + \frac{1}{H \sqrt{1 + |Du(x)|^2}} \right] \geq \max_{\partial \Omega} u.$$  

Having established this, if $p = (x_0, u(x_0))$ is a non-umbilic point, then we have an interior local maximum for a graph with positive constant mean curvature $-H$. That is an immediate contradiction of the comparison principle.

We are left with one final case: $p$ is an umbilic point. In this singular case, the set $\mathcal{S} = \{X + N/H : X \in \mathcal{S}\}$ may not be a regular surface. Nevertheless, let us change variables and express $\mathcal{S}$ locally as a graph $(\bar{x}, \bar{u})$ over $T_p \mathcal{S}$ with $\bar{p} = 0$ and $\bar{q} = (0, 1/H)$ the point in $\mathcal{S}$ with maximum third component in the old coordinates, see Figure 1. More generally, points in $\mathcal{S}$ are given by

$$Y(\bar{x}) = (\bar{x}, \bar{u}) + \frac{(-\bar{u}_x, -\bar{u}_y, 1)}{H \sqrt{1 + |Du|^2}}.$$  

Technically, $\bar{x} = (\bar{x}, \bar{y})$ and when we write $\bar{u}_x$, we really mean $\bar{u}_x$, but we are going to leave those bars off the $x$’s and $y$’s especially when the $u$ already has a bar on it. Just remember we’re working in the new bar-coordinates for the moment.

We introduce, furthermore, the spherical cap in the same coordinates given by

$$v(\bar{x}) = \frac{1}{H} + \sqrt{\frac{1}{H} - |\bar{x}|^2} = \frac{1}{H} \left( 1 - \sqrt{1 - H^2|\bar{x}|^2} \right).$$  

The key in this singular case is the following fact which we state in the special case at hand, but some form of which holds in general for pairs of solutions of second order elliptic equations:
Theorem 3 Since $\bar{u}$ and $v$ agree up to second order at the origin, then either

(a) the difference $\bar{u} - v \equiv 0$, or

(b) $p$ is an isolated umbilic and the difference is of the form

$$\bar{u} - v = h + O(|\bar{x}|^{k+1})$$

where $h$ is a harmonic polynomial of finite order $k \geq 3$.

Harmonic Polynomials

It’s easiest to describe these in complex notation, and we will also suppress the bars. If $z = x + iy$, then a harmonic polynomial of degree $k$ is a function of the form

$$h(x) = \text{Re}(z_0 z^k)$$

where $z_0$ is a complex constant. This multiplication by a constant only rotates and dilates the graph of $\text{Re}(z^k)$ which is easily seen to have $2k$ sign changes in a neighborhood of the origin. For example, $x$ is the basic order one harmonic polynomial, while $x^2 - y^2$ is the basic second order harmonic polynomial. The graph of $\text{Re}(z^3)$ has six sign changes and is shown in Figure 2.

Returning to the proof, if $\bar{u} - v \equiv 0$, then $S$ is locally spherical.

Exercise 12 Show that the set of umbilic points is both open and closed in $\Omega$ and hence is either empty or all of $\Omega$. Hint: Use the theorem on harmonic polynomials above.

Thus, we may assume our particular umbilic is isolated, we have the representation of $\bar{u}$ in terms of the spherical cap $v$ and the harmonic polynomial $h$, and we can proceed to consider expansions as follows.
Exercise 13  

1. Use the Taylor expansion $(1 + \xi)^{-1/2} = 1 - \xi/2 + O(|\xi|^2)$ to show
\[
\frac{1}{\sqrt{1 + |Du|^2}} = \frac{1}{\sqrt{1 + |Du|^2}}[1 - Dv \cdot Dh + O(|\bar{x}|^{k+1})]
\]
\[
= \sqrt{1 - \mu^2|\bar{x}|^2} - HDh \cdot \bar{x} + O(|\bar{x}|^{k+1}).
\]

2. Conclude that
\[
Y_1(\bar{x}) = -h_x/H + O(|\bar{x}|^k), \quad Y_2(\bar{x}) = -h_y/H + O(|\bar{x}|^k),
\]
and
\[
Y_3(\bar{x}) = \frac{1}{H} + h - Dh \cdot \bar{x} + O(|\bar{x}|^{k+1}).
\]

3. Use the expression $h(\bar{x}) = \text{Re}(z_0z^k)$ to obtain
\[
Y_1 - iY_2 = -kz_0z^{k-1}/H + O(|\bar{x}|^k)
\]
and
\[
Y_3(\bar{x}) = \frac{1}{H} + (1 - k)\text{Re}(z_0z^k) + O(|\bar{x}|^{k+1}).
\]

4. Conclude that since
\[
\sqrt{Y_1^2 + Y_2^2} = c|\bar{x}|^{k-1} + O(|\bar{x}|^k)
\]
and
\[
Y_3(\bar{x}) = d|\bar{x}|^k + O(|\bar{x}|^{k+1}),
\]
$\bar{\mathcal{S}}$ has a well defined tangent plane at $\bar{q}$.
Notice that if $Du(p_1, p_2) \neq 0$, then the tangent plane and hence points of $\mathcal{S}$ extend both below and above the plane $z = q_3$ which contradicts the assumption that $q_3$ is maximal.

Furthermore, even if $Du(p_1, p_2) = 0 = D\tilde{u}(0)$, then $\mathcal{S}$ behaves in a neighborhood of $q = x + e_3/H$ like the graph of a harmonic polynomial with points both above and below the plane $z = q_3$. This is, of course, the same contradiction.

This completes Serrin’s proof.

We now sketch quickly a second proof of Serrin’s result which avoids some of the technicalities (or at least packages them differently). This proof is also based on the maximum principle, but instead of using the mean curvature operator which operates on smooth functions on $\Omega$, we consider a particular elliptic operator which operates directly on smooth functions on the CMC surface. This operator is the intrinsic Laplacian or Laplace-Beltrami operator. We will not explain in detail how this operator works or how to compute it. We only list several important facts about it.

Let $\mathcal{F}(\mathcal{S})$ denote the collection of smooth real valued functions on $\mathcal{S}$. The first important fact is that the intrinsic Laplacian is an operator $\Delta^\mathcal{S} : \mathcal{F}(\mathcal{S}) \to \mathcal{F}(\mathcal{S})$ such that a comparison principle holds:

**Theorem 4** If $u, v \in \mathcal{F}(\mathcal{S}) \cap C^0(\mathcal{S})$ and

\[
\begin{align*}
\Delta^\mathcal{S} u &\leq \Delta^\mathcal{S} v \quad \text{on } \mathcal{S} \\
u\big|_{\partial\mathcal{S}} &\geq v\big|_{\partial\mathcal{S}},
\end{align*}
\]

then $u \geq v$ on $\mathcal{S}$ with equality only if $u \equiv v$. Note that here we have returned to the usual notation of using a bar for the closure: $\mathcal{S} = \mathcal{S} \cup \partial\mathcal{S}$.

On a surface $\mathcal{S}$ of constant mean curvature, all the coordinate functions of $X$ and $N$ are in $\mathcal{F}(\mathcal{S})$.

**Lemma 4**

\[
\Delta^\mathcal{S} X = 2HN = (\kappa_1 + \kappa_2)N;
\]

and

\[
\Delta^\mathcal{S} N = -(\kappa_1^2 + \kappa_2^2)N,
\]

**Exercise 14** Use the lemma above to show that

\[
\Delta^\mathcal{S} \left( X_3 + \frac{1}{H}N_3 \right) = -\frac{(\kappa_1 - \kappa_2)^2}{H}N_3.
\]

**Exercise 15** Conclude from the maximum principle that if $\mathcal{S}$ is the graph of a function $u \in C^2(\Omega)$ and $N$ is the upward normal, then

\[
X_3 + \frac{1}{H}N_3 = u + \frac{1}{H}N_3 \leq \max_{\partial\Omega} u.
\]

Conclude that Serrin’s estimate holds.
We have not considered the case of gradient blowup nor of equality, but those conclusions of Serrin’s height estimate can also be obtained fairly easily via this second approach. For now, let us return to the general course of apriori estimates and outline some more serious bootstrapping.

**Apriori estimates for the gradient**

In the discussion of Serrin’s height estimate, we have worried ourselves a little with the possibility of gradient blow-up on $\partial \Omega$, i.e., considering solutions $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. We now return to our more leisurely assumption that $u \in C^\infty(\Omega) \cap C^2(\overline{\Omega})$.

**Theorem 5**

\[
\max_\Omega |Du| = \max_{\partial \Omega} |Du|.
\]  

**Proof:** Set $v = |Du|^2$. Evidently,

\[
v|_{\partial \Omega} \leq (\max_{\partial \Omega} |Du|)^2.
\]

Thus, if we can find a linear elliptic operator $L$, to which the maximum principle applies, and for which $Lv \geq 0$, then we will have established (5).

We begin by differentiating the equation with respect to $x_k$:

\[
\sum_{i,j} \left[ A_{ij} (Du) D_i D_j (D_k u) + \sum_{\ell} \frac{\partial A_{ij}}{\partial p_\ell} (Du) D_\ell D_k u D_i D_j u \right] = 0.
\]

Next, multiplying by $D_k u$ and summing over $k$ we find

\[
0 = \sum_{i,j,k} A_{ij} (Du) D_i D_j (D_k u) D_k u + \sum_{i,j,k,\ell} \frac{\partial A_{ij}}{\partial p_\ell} (Du) D_\ell D_k u D_i D_j u D_k u. \tag{6}
\]

On the other hand,

\[
\sum_{i,j} A_{ij} (Du) D_i D_j v = \sum_{i,j,k} A_{ij} (Du) D_i (2D_k v D_j D_k v)
\]

\[
= 2 \sum_{i,j,k} A_{ij} (Du) (D_i D_k v D_j D_k v + D_k v D_i D_j D_k v).
\]

Multiplying (6) by 2 and subtracting we find

\[
\sum_{i,j} A_{ij} (Du) D_i D_j v = 2 \sum_{i,j,k} A_{ij} (Du) D_i D_k v D_j D_k v - 2 \sum_{i,j,k,\ell} \frac{\partial A_{ij}}{\partial p_\ell} (Du) D_\ell D_k u D_i D_j u D_k u. \tag{7}
\]
Considering finally that
\[ D_\ell v = 2 \sum_k D_k u D_\ell D_k u, \]
and comparing to the last term in (7), we see that
\[ \sum_{i,j,\ell} \left( \frac{\partial A_{ij}}{\partial p_\ell} (D_u) D_i D_j u \right) D_\ell v = 2 \sum_{i,j,k,\ell} \frac{\partial A_{ij}}{\partial p_\ell} (D_u) D_i D_k u D_j D_k u. \]
Thus, using ellipticity
\[ \sum_{i,j} A_{ij} (D_u) D_i D_j v + \sum_{i,j,\ell} \left( \frac{\partial A_{ij}}{\partial p_\ell} (D_u) D_i D_j u \right) D_\ell v = 2 \sum_{i,j,k} A_{ij} (D_u) D_i D_k v D_j D_k v \]
\[ \geq 2 \sum_k \epsilon_0 |D D_k v|^2 \]
\[ \geq 0. \]
We have then a linear operator \( L \) for which
\[ Lv = \sum a_{ij}(x) D_i D_j v + \sum b_\ell(x) D_\ell v \geq 0. \]
It turns out that the first order terms \( \sum b_\ell D_\ell v \) do not cause a problem in the proof of the E. Hopf strong maximum principle and, therefore, we have established (5). \( \square \)

As the hemispherical graph indicates, we cannot always expect \( \sup_{\partial \Omega} |D u| \) to be finite valued. Indeed, this is the point at which the interplay between the boundary values \( \phi \) and the shape of \( \Omega \) plays the key role in determining the solvability of the problem. Our task, in this instance, is significantly simplified if we assume zero boundary values and a convex domain. Since our primary case of interest has these features, we do not hesitate to make those assumptions. Accordingly, we prove the following apriori boundary gradient estimate:

**Theorem 6** If \( u_0 \in C^2(\bar{\Omega}) \) has graph with constant mean curvature \( H_0 \) and zero boundary values on the convex \( C^{2,\alpha} \) domain \( \Omega \), then there is some \( \epsilon > 0 \) and a constant \( M > 0 \) such that
\[ \max_{\partial \Omega} |D u| \leq M \]
for any solution \( u \) of
\[ \begin{cases} 
\text{div} \left( \frac{D u}{\sqrt{1 + |D u|^2}} \right) = 2H & \text{on } \Omega \\
u \big|_{\partial \Omega} = 0 \end{cases} \]
with \( |H - H_0| < \epsilon \).
Proof: We will use the comparison principle directly to get a gradient estimate on the boundary. More precisely, we look for a function $w$ which satisfies

$$\begin{align*}
\mathcal{M}w &\leq 2H_1 < 0 \quad \text{on } \Omega \\
u |_{\partial \Omega} &= 0.
\end{align*}$$

Checking the comparison principle, you will find that any solution $u$ with CMC $H > H_1$ must satisfy $u \leq w$. Consequently, on $\partial \Omega$

$$D_n u \leq D_n w$$

where $n$ is the unit inward normal to $\partial \Omega$. This the the boundary gradient estimate from one direction since $|Du| = |D_n u|$. A function used in this way is called a barrier. Evidently, we only need to consider $H_0 \leq 0$.

Now, we will use a little believable regularity which we will not completely justify. Remember that we are starting with a solution $u_0$ given.

Consider the linear operator

$$\mathcal{L}w = \sum_{i,j} A_{ij}(Du_0)D_iD_jw + \sum_k \frac{\partial A_{ij}}{\partial p_k}(Du_0)D_kw.$$

**Exercise 16** Show that $\mathcal{L}$ is the linearization of the mean curvature operator $\mathcal{M}$, i.e., think of $\mathcal{M}u$ as a function $F(Du, D^2u)$ where $F = F(p, Q) : \mathbb{R}^2 \times \mathbb{R}^{2 \times 2} \to \mathbb{R}$, and find the first order part of the Taylor expansion of $F$ at $(p_0, Q_0) = (Du_0, D^2u_0)$.

Next define $w = w(x; H)$ to be the solution of the linear boundary value problem

$$\begin{align*}
\mathcal{L}w &= 2H \quad \text{on } \Omega \\
w |_{\partial \Omega} &= 0.
\end{align*}$$

It turns out that $w$ depends smoothly on $H$; in fact, $w \in C^\infty(\Omega \times (-\infty, \infty)) \cap C^2(\bar{\Omega} \times (-\infty, \infty))$. Denoting derivatives with respect to $H$ by using a “dot” as in

$$\frac{\partial w}{\partial H} = \dot{w},$$

we find

$$\left. \frac{\partial}{\partial H} \mathcal{M}w \right|_{H=H_0} = \mathcal{L}\dot{w} \big|_{H=H_0} = 2.$$

By uniform continuity on the compact set $\bar{\Omega}$, we have that for some $H_1 < H_0$

$$\frac{\partial}{\partial H} \mathcal{M}w > 1 \quad (8)$$

12
for all $x \in \Omega$ and $H \in (H_1, H_0)$. In particular, it cannot be the case that $\mathcal{M}w \geq 2H_0$ for $H \in (H_1, H_0)$ since then we would have

$$\mathcal{M}w(x; H_0) - \mathcal{M}w(x; H)/(H_0 - H) = 2H_0 - \mathcal{M}w(x; H)/(H_0 - H) \leq 0,$$
and this would contradict (8) by the mean value theorem. Thus, taking any particular, $H_2 \in (H_1, H_0)$, we have

$$\mathcal{M}w(x; H_2) \leq \max_{\xi \in \Omega} \mathcal{M}w(\xi; H_2) = 2(H_0 - \epsilon) < 2H_0.$$

Evidently, $w = w(x; H_2)$ provides a barrier which implies the result. □

**Exercise 17** Give an alternative construction of a barrier for any boundary point on a convex domain which works as long as $|H_0| < \sqrt{3}/(2R_{out})$ where $R_{out}$ is the radius of the smallest disk containing $\Omega$. Hint: Use the height estimate and horizontal quarter cylinders.

**Exercise 18** Find the linearized operator $L$ when $H_0 = 0$. Show explicitly why the barrier works in this case, i.e., write down $\mathcal{M}w$ and explain why this expression is negative.

### Hölder Estimates for the Gradient

In view of Theorem 5 and Theorem 6, if we have existence of solutions $u = u(x; H)$ for $|H| \leq H_0$, then all such solutions satisfy (uniform) height and gradient bounds

$$|u(x; H)| \leq \max_{\Omega} |u(x; H_0)|$$

$$|Du(x; H)| \leq \max_{\partial\Omega} |Du(x; H_0)|.$$

Evidently, this situation will persist (if existence persists) until we reach some “largest” $H_0$ for which

$$\sup_{|H| < H_0} \sup_{\partial\Omega} |Du(x; H)| = \infty.$$

**Exercise 19** Show that there must be some such largest $H_0$. Hint: Integrate the equation and apply the divergence theorem to conclude that

$$2|H| \leq \frac{\text{length}(\partial\Omega)}{\text{area}(\Omega)}.$$

Can you think of a domain for which this bound is sharp?

Having said all that, we still do not have existence according to the Leray-Schauder fixed point theorem. We still need to get one more apriori estimate:
Theorem 7 If $u$ is a solution of the CMC equation with zero boundary values, then for each $k = 1, 2$

$$[D_k u]_\alpha \leq M$$

where $M$ is a constant depending only on $H$ and $\Omega$, but not on $u$.

The details of the proof of this theorem are quite involved, and we will content ourselves with making a few comments about them—without actually doing them.

First of all, it should be mentioned that this Hölder estimate has little to do with the relation between the geometry, i.e., shape, of $\Omega$ and the solvability of the problem. That relation is captured in the gradient estimates above. The Hölder estimate for $Du$ holds under much more general conditions having nothing to do with mean curvature or even quasilinear equations. Nevertheless, this basic story is worth understanding.

Bounded Measureable Coefficients?

Let us review our situation. We have an equation

$$\sum A_{ij}(Du)D_iD_ju = 2H.$$ 

We need apriori $C^{1,\alpha}$ estimates to solve this equation. We can do the $C^1$ part. If we think of this as a linear equation, therefore, we could get good estimates using Linear Schauder theory... if we had some estimates on the coefficients $a_{ij}(x) = A_{ij}(Du(x))$. In fact, we would only need a $C^{0,\alpha}$ estimate for the $a_{ij}$. But we do not have that; we only have a $C^0$ estimate. This situation leads to the following (unexpected) question:

What can you say about solutions of a linear PDE if you only know the coefficients are bounded?

So people thought about this for a long time, and eventually began to dwell on weak solutions for elliptic equations. We will not embark on the theory of weak solutions, but let it suffice to say

1. It is natural to assume some kind of divergence form for the equation in the theory of weak solutions,

2. It is natural to assume the coefficients are bounded and measurable, and

3. Integration and $L^p$ estimates play a much more important role in this theory than in the material we have chosen to cover.

Here is what they came up with:
Theorem 8 (The DiGiorgi-Nash-Moser Theorem) If you have a (weak) solution $u$ of a linear elliptic equation of the form
\[ \sum_{i,j} D_i [a_{ij}(x)D_ju] = 0, \]
then $u \in C^{0,\alpha}$ for some $\alpha$ and you get an apriori estimate
\[ [u]_\alpha \leq M \]
where the constant $M$ depends only on $\Omega$, the ellipticity constant for the $a_{ij}$ and their $C^0$ norm, but not on $u$.

The coefficients are actually allowed to be only bounded and measurable, so the $C^0$ norm should really be replaced by the essential sup norm of $L^\infty$.

The thing that I would like for you to remember is this: One of these days when you study the details of the linear theory of elliptic PDE, you will expend a considerable amount of time and energy understanding linear equations with bounded measurable coefficients. When this happens, remember that we don’t really care about linear equations with bounded measurable coefficients; we care about nonlinear equations (especially quasilinear equations), and it just so happens that we don’t have any estimates on their coefficients...so we’re forced to this.

Final Bootstrapping

Once you have the DiGiorgi-Nash-Moser Theorem, here’s the clever variant on bootstrapping which allows you to use it. This idea is due to Ladyzhanskaia and Uraltsseva. Differentiate the equation with respect to $x_k$ taking note of its divergence form as follows:
\[ \sum_{i,j} D_i \left[ \frac{\partial A_i}{\partial p_j}(Du) D_j(D_ku) \right] = 0 \]
where
\[ A_1 = \frac{u_x}{\sqrt{1 + |Du|^2}} \quad \text{and} \quad A_2 = \frac{u_y}{\sqrt{1 + |Du|^2}} \]
Notice that we know the coefficients in this equation are bounded. Thus, by the DiGiorgi-Nash-Moser Theorem,
\[ [D_ku]_\alpha \leq M, \]
and we are done.

In summary and in review:
1. Comparison implies $|Du|$ is apriori bounded on $\partial \Omega$.

2. The Maximum Principle implies $|Du|$ is apriori bounded on $\Omega$.

3. $D_k u$ satisfies a divergence form PDE with bounded coefficients. This implies $[D_k u]_\alpha$ is apriori bounded.

4. These apriori bounds are what are required by the Leray-Schauder Fixed Point Theorem.