Assignment 1 Problem 7
Solution

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Here is the statement of Problem 7 of Assignment 1:

> *Check, as claimed in the Lecture 1 notes, that the image of the horizontal segment* $\{x+iy : -\pi/2 < x < \pi/2\}$ *(for* y *fixed) under the complex tangent function lies on a circle with center at* $i\coth(2y)$ *on the imaginary axis.*

Here is a detailed solution: We need to compute $|\tan z - i\coth(2y)|$ where $z = x+iy$, and show that the resulting expression is independent of x, i.e., depends only on y. If we can do this, we will have shown that $\tan z$ lies on a circle for y fixed, and the value $r = r(y)$ we obtain will be the radius of the circle. As a side note, this problem could have been stated without giving the center of the circle. In that case, we could conclude from the symmetry of the complex tangent that the circle would have to be centered on the imaginary axis (as also suggested by my Mathematica plots), and then we could come up with a conjectured center. In the next problem, Problem 8, you are not told that the image is a circle or anything else, so you will have to come up with your own conjecture (and then try to verify it).

Let $M = \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y$. We have already calculated the real and imaginary parts of

$$\tan z = \frac{\cos x\sin x}{M} + i\,\frac{\cosh y\sinh y}{M}.$$

Here is an important observation/trick/strategy: Whenever you have an expression, like M, involving only even powers of the basic trigonometric functions $\cos x$ and $\sin x$ and the basic hyperbolic trigonometric functions $\cosh y$ and $\sinh y$, then you can use the identities $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$ to express the value as a **unique polynomial** in $\cos x$ and $\cosh y$. This can be helpful if you are wanting to recognize particular values within certain expressions. This is our basic strategy below.

Applying this observation to M, we see

$$\begin{aligned} M &= \cos^2 x \cosh^2 y + (1 - \cos^2 x)(\cosh^2 y - 1) \\ &= \cos^2 x \cosh^2 y + \cosh^2 y - 1 - \cos^2 x \cosh^2 y + \cos^2 x \\ &= \cos^2 x + \cosh^2 y - 1. \end{aligned}$$

Not only is this a unique polynomial expression which can be identified elsewhere, it is fairly simple. There may be other alternatives. For example, we could also note that

$$\cos(2x) + \cosh(2y) = 2\cos^2 x + 2\cosh^2 y - 2,$$

so

$$M = \frac{1}{2}[\cos(2x) + \cosh(2y)].$$

We won't use this expression, but noting that we can also write

$$\tan z = \frac{1}{2M}[\sin(2x) + i \sinh(2y)]$$

one suspects there is a slick solution of this problem all in terms of double angles/arguments. I am not trying to present the slickest solution. I'll leave that to you.

I'm going to compute the square of the expression $|\tan z - i \coth(2y)|$. Recall that

$$\coth(2y) = \frac{\cosh^2 y + \sinh^2 y}{2\cosh y \sinh y} = \frac{2\cosh^2 y - 1}{2\cosh y \sinh y}.$$

In the last expression we don't just get cosine and hyperbolic cosine because we do not have only quadratic powers. Nevertheless,

$$\begin{aligned} |\tan z - i\coth(2y)|^2 &= \left| \frac{\cos x \sin x}{M} + i\left(\frac{\cosh y \sinh y}{M} - \frac{\cosh^2 y + \sinh^2 y}{2\cosh y \sinh y} \right) \right|^2 \\ &= \frac{\cos^2 x \sin^2 x}{M^2} + \left(\frac{\cosh y \sinh y}{M} - \frac{\cosh^2 y + \sinh^2 y}{2\cosh y \sinh y} \right)^2 \\ &= \frac{\cos^2 x \sin^2 x}{M^2} + \frac{\cosh^2 y \sinh^2 y}{M^2} - \frac{\cosh^2 y + \sinh^2 y}{M} \\ &\qquad + \left(\frac{\cosh^2 y + \sinh^2 y}{2\cosh y \sinh y} \right)^2. \end{aligned}$$

Notice that the last term is independent of x. Thus, we focus on the first three terms:

$$\frac{1}{M^2}[\cos^2 x - \cos^4 x + \cosh^4 y - \cosh^2 y - M(2\cosh^2 y - 1)].$$

The expression $M(2\cosh^2 y - 1)$ expands as follows:

$$(2\cosh^2 y - 1)(\cos^2 x + \cosh^2 y - 1) = 2\cos^2 x \cosh^2 y + 2\cosh^4 y - 3\cosh^2 y - \cos^2 x + 1.$$

Therefore,

$$\begin{aligned} \cos^2 x - \cos^4 x + \cosh^4 y - \cosh^2 y - M(2\cosh^2 y - 1) = \\ 2\cos^2 x - \cos^4 x - \cosh^4 y + 2\cosh^2 y - 2\cos^2 x \cosh^2 y - 1. \end{aligned}$$

We expect to cancel a factor of M^2 from this expression. And perhaps now you can see it:

$$M^2 = \cos^4 x + 2\cos^2 x \cosh^2 y - 2\cos^2 x + \cosh^4 y - 2\cosh^2 y + 1.$$

That is, the numerator from the first three terms of $|\tan z - i\coth(2y)|^2$ is

$$\cos^2 x - \cos^4 x + \cosh^4 y - \cosh^2 y - M(2\cosh^2 y - 1) = -M^2.$$

Therefore,

$$|\tan z - i\coth(2y)|^2 = -1 + \coth^2(2y) = \frac{1}{\sinh^2(2y)} = \operatorname{csch}^2(2y).$$

We have verified the assertion of the problem, and the radius is $r = |\operatorname{csch}(2y)|$. $\square$