Lecture 1: Complex Numbers
Assignment Problems Due Friday August 28, 2020

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Problem 1 *The* **complex conjugate** *of a complex number* $z = x + iy$ *is* $\bar{z} = x - iy$. *Find* $z + \bar{z}$, $z - \bar{z}$, *and* $z\bar{z}$. *Express* $\text{Re}(z)$, $\text{Im}(z)$, *and* $|z|$ *in terms of* z *and* $\bar{z}$.

Problem 2 *Draw the following subsets in the complex plane.*

(a) *(Boas 2.4)* $\{z \in \mathbb{C} : |z - 2| = 1\}$.

(b) *(Boas 2.5.46)* $\{z \in \mathbb{C} : z = -i\bar{z}\}$.

(b) *(Boas 2.5.52)* $\{z \in \mathbb{C} : \text{Re}\, z = 1\}$.

Problem 3 *Check that the series expansions*

$$\cos z = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell z^{2\ell}}{(2\ell)!} \qquad \text{and} \qquad \sin z = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell z^{2\ell+1}}{(2\ell+1)!}$$

hold for $z \in \mathbb{C}$ *using the definitions*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and the series expansion (definition) for the exponential.

Problem 4 *Show that if one defines cosine and sine for complex arguments using the series expansions*

$$\cos z = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell z^{2\ell}}{(2\ell)!} \qquad \text{and} \qquad \sin z = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell z^{2\ell+1}}{(2\ell+1)!},$$

then one can prove the formulas

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Problem 5 *Check that for $z = x + iy$*

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

using the definitions

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

for the real hyperbolic cosine and sine.

Problem 6 *Plot the real hyperbolic cosine and sine. Derive the real McClauren series expansions for* $\cosh x$ *and* $\sinh x$ *and determine the radii of convergence for these series.*

Problem 7 *Check, as claimed in the Lecture 1 notes, that the image of the horizontal segment* $\{x + iy : -\pi/2 < x < \pi/2\}$ *(for y fixed) under the complex tangent function lies on a circle with center at* $i \coth(2y)$ *on the imaginary axis.*

Problem 8 *Determine the image of a vertical line* $\{x + iy \in \mathbb{C} : y \in \mathbb{R}\}$ *(with x fixed) under the complex tangent function.*

Problem 9 *Express* $\text{Arg}(z)$ *properly in terms of a branch of the complex inverse tangent function.*

Problem 10 *Determine the value of the complex logarithm and determine the associated Riemann surface by understanding the complex exponential as a mapping.*