Assignment 2 Problem 1
Solution

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Here is the statement of Problem 1 of Assignment 2:

> *(Boas 2.7.2) Discuss the domain of convergence of the complex alternating harmonic power series*
>
> $$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

Here is a detailed solution/discussion: The first thing to realize is that **every power series has a radius/disk of convergence** associated[1] with it. The center of expansion (in this case $z = 0$) tells you the center of the disk. The radius r is determined by the following:

> *If $|z| < r$, then the series converges absolutely, i.e., the series with terms given by the absolute values of the terms in "your" series converges; if $|z| > r$, then the series diverges.*

As we consider this series

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots = \sum_{j=0}^{\infty} (-1)^j \frac{z^{j+1}}{j+1},$$

the associated series of absolute values is

$$\sum_{j=0}^{\infty} \frac{|z|^{j+1}}{j+1}.$$

[1] When I type things in boldface here, I'm talking about an important concept which you would do well to know about or might want to look up and study/review.

This is, of course, a **series of positive terms**, and series of positive terms are either bounded above (and convergent) or not bounded above (and divergent). This follows from the fact that the **sequence of partial sums** is non-decreasing, and consequently series of positive terms are much easier to deal with.

The series of absolute values is bounded above by the **geometric series**

$$\sum_{j=0}^{\infty} |z|^{j+1} = -1 + \sum_{j=0}^{\infty} |z|^{j} = -1 + \sum_{j=0}^{\infty} \rho^{j}$$

where $\rho = |z|$ is called the **ratio** for the geomtric series. The partial sum

$$\sum_{j=0}^{k} \rho^{j}$$

is given explicitly by

$$\sum_{j=0}^{k} \rho^{j} = \frac{1-\rho^{k}}{1-\rho}.$$

(To see this, just multiply out the product

$$(1-\rho)\sum_{j=0}^{k} \rho^{j}$$

using the distributive property and cancel terms.) The sequence

$$\left\{\frac{1-\rho^{k}}{1-\rho}\right\}_{k=1}^{\infty} \tag{1}$$

has a finite limit

$$\sum_{j=0}^{\infty} \rho^{j} \frac{1}{1-\rho}$$

when $0 \leq \rho < 1$. We conclude that our original series **converges absolutely** for $|z| < 1$.

Exercise 1 *What happens to the sequence of partial sums when $\rho > 1$? What happens to the geometric series*

$$\sum_{j=0}^{\infty} \rho^{j}$$

when $\rho = 1$?

By the main fact concerning the disk of convergence for power series stated above we know our series converges absolutely for $|z| < 1$. Also, to show the radius of convergence is $r = 1$, it is enough to find one point z with $|z| = 1$ where the series does not converge. Take $z = -1$. Then our original series becomes

$$\sum_{j=0}^{\infty}(-1)^j \frac{(-1)^{j+1}}{j+1} = -\sum_{j=0}^{\infty} \frac{1}{j+1}.$$

The series

$$\sum_{j=0}^{\infty} \frac{1}{j+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

is called the **harmonic series** and is known to diverge to $+\infty$. More precisely, this is a series of positive terms with unbounded sequence of partial sums. To see this one can group terms as follows:

$$1+\frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$
$$+ \left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}\right).$$

That is, the partial sum

$$\sum_{j=1}^{2^{k+1}} \frac{1}{j}$$

can be written as

$$\sum_{j=1}^{2^{k+1}} \frac{1}{j} = 1 + \sum_{\ell=0}^{k} \sum_{m=2^\ell+1}^{2^{\ell+1}} \frac{1}{m}.$$

Since

$$\sum_{m=2^\ell+1}^{2^{\ell+1}} \frac{1}{m} \geq \sum_{m=2^\ell+1}^{2^{\ell+1}} \frac{1}{2^{\ell+1}} = \left(2^{\ell+1} + 1 - (2^\ell + 1)\right) \frac{1}{2^{\ell+1}} = \frac{1}{2},$$

this means

$$\sum_{j=1}^{2^{k+1}} \frac{1}{j} = 1 + \sum_{\ell=0}^{k} \frac{1}{2} = 1 + \frac{k+1}{2} \to \infty \qquad \text{as } k \to \infty.$$

Exercise 2 *The particular grouping above (in which the partial sum is seen to have groupings with sum at least* $1/2$*) was suggested by Chris Page in office hours. Other groupings to show the harmonic series diverges are possible. Work out the details to obtain groupings (starting with the first term 1) so that each group in the partial sum has sum 1. Hint: Show that starting with any term* $1/m$ *it is possible to add additional terms* $1/m+1, \ldots 1/M$ *such that the sum*

$$\frac{1}{m} + \cdots \frac{1}{M} > 1.$$

At this point, we know the radius of convergence is $r = 1$. If you look at the statement of Problem 2.7.2 in Boas, you will see that she only asks for the disk of convergence. When you see a problem statement like mine which says "Discuss..." then, of course, you're looking at something more open ended. That is to say, we could stop here, but if we go on things are going to get more difficult.

1 Convergence when $|z| = 1$.

We know our power series diverges for $|z| > 1$ and converges (absolutely) for $|z| < 1$. It remains to consider points on the boundary of the unit disk. We know there is divergence to $-\infty$ when $z = -1$. The next step might be to consider $z = 1$. In this case, our series becomes

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{j+1}.$$

This is an **alternating** real series with terms tending to zero, and it is known that all such series converge. To see this, it is enough to note that the partial sum

$$s_k = \sum_{j=0}^{k} \frac{(-1)^j}{j+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^k}{k+1}$$

satisfies

$$s_{k-1} < \sum_{j=0}^{k} \frac{(-1)^j}{j+1} < s_{k-2} \qquad \text{if } k \text{ is even}$$

and

$$s_{k-2} < \sum_{j=0}^{k} \frac{(-1)^j}{j+1} < s_{k-1} \qquad \text{if } k \text{ is odd.}$$

It follows inductively that the sequence $\{s_{2\ell}\}_{\ell=0}^{\infty}$ of partial sums with even indices is a decreasing sequence which is bounded below by each partial sum with an odd index. Similarly, the sequence $\{s_{2\ell+1}\}_{\ell=0}^{\infty}$ of partial sums with odd indices is an increasing sequence which is bounded above by each partial sum with an even index. Since

$$s_{2\ell} - s_{2\ell+1} = \frac{1}{2(\ell+1)} \to 0 \qquad \text{as } \ell \to 0,$$

we conclude the sequence of partial sums, and hence the alternating series, converges.

Exercise 3 *Show that the alternating harmonic power series also converges at* $z = \pm i$.

Breandan Yeats was interested in understanding what happens at the other points of the unit circle, and eventually we were led[2] to Example 3.40 on page 69 of Walter Rudin's classic text *Principles of Mathematical Analysis.* Rudin shows there that the series we are considering converges at all points with $|z| = 1$ except $z = -1$. Rudin's proof applies to our series as follows:

Write

$$\sum_{j=0}^{\infty} (-1)^j \frac{z^{j+1}}{j+1} = -\sum_{j=0}^{\infty} \frac{(-z)^{j+1}}{j+1}.$$

Note that the partial sums

$$\sum_{j=0}^{k} (-z)^{j+1}$$

are **bounded** as long as $z \neq -1$ because

$$\left| \sum_{j=0}^{k} (-z)^{j+1} \right| = \left| \frac{1-(-z)^k}{1-(-z)} \right| \leq \frac{2}{|1+z|}.$$

Notice that the bound $M = 2/|1+z|$ is a fixed positive number independent of k as long as z is fixed with $|z| = 1$ and $z \neq -1$.

Now, we are going to use another fact, which is relatively easy to prove (and you can look up):

[2] See the section "Notes" at the end of this solution.

A series of complex numbers $\sum a_j$ converges if and only if the corresponding sequence of partial sums is **Cauchy***, i.e., for any $\epsilon > 0$, there is some index N such that*

$$\left|\sum_{j=1}^{k} a_j - \sum_{j=1}^{\ell} a_j\right| < \epsilon \qquad \textit{whenever } k, \ell > N.$$

The **Cauchy condition** for a sequence basically says that the terms in the sequence "bunch up" out at the end. If a sequence is Cauchy, it is likely to converge, and for sequences of complex numbers it will converge.

We will also use Rudin's clever **partial summation formula**, or rearrangement of terms, which in our case takes the form

$$\sum_{j=\ell+1}^{k} \frac{(-z)^j}{j} = -\frac{1}{\ell+1}\sum_{j=1}^{\ell}(-z)^j + \sum_{m=\ell+1}^{k-1}\left(\frac{1}{m} - \frac{1}{m+1}\right)\sum_{j=1}^{m}(-z)^j + \frac{1}{k}\sum_{j=1}^{k}(-z)^j$$

where $\ell < k$. To see Rudin's identity, write $(-z)^m$ as

$$(-z)^m = \sum_{j=1}^{m}(-z)^j - \sum_{j=1}^{m-1}(-z)^j.$$

Multiply both sides by $1/m$ and sum (both sides) from $m = \ell + 1$ to $m = k$. You should get two double sums on the right. If you shift the indices m in the second sum, you can get Rudin's rearrangement.

Exercise 4 *Write out the details.*

Finally then, we attempt to verify the Cauchy condition for our sequence of partial sums: Let $\epsilon > 0$ and recall that $M = 2/|1 + z|$. (Here we are still assuming, of course, that z is fixed with $|z| = 1$ and $z \neq -1$.) We can find some N for which

$1/N < \epsilon/(2M)$. Then if $k, \ell > N$, we can assume $k < \ell$ and we have

$$\begin{aligned}
\left|\sum_{j=\ell+1}^{k} \frac{(-z)^j}{j}\right| &= \left|-\frac{1}{\ell+1}\sum_{j=1}^{\ell}(-z)^j + \sum_{m=\ell+1}^{k-1}\left(\frac{1}{m}-\frac{1}{m+1}\right)\sum_{j=1}^{m}(-z)^j + \frac{1}{k}\sum_{j=1}^{k}(-z)^j\right| \\
&\le M\left|\frac{1}{\ell+1} + \sum_{m=\ell+1}^{k-1}\left(\frac{1}{m}-\frac{1}{m+1}\right) + \frac{1}{k}\right| \\
&= \frac{2M}{\ell+1} \\
&< \frac{2M}{N} \\
&< \epsilon.
\end{aligned}$$

This shows that the alternating harmonic power series converges for every z on the unit circle in $\mathbb{C}$ except $z = -1$. This settles the question of convergence versus divergence. However, one may also be interested in the actual values taken by the series as well.

Exercise 5 *Use Rudin's approach to prove his Theorem 3.44: If the radius of convergence of $\sum a_j z^j$ is $r = 1$ and the coefficients decrease monotonically to zero satisfying*

$$a_0 \ge a_1 \ge a_2 \ge \cdots \qquad \text{with} \qquad \lim_{j\nearrow\infty} a_j = 0,$$

then $\sum a_j z^j$ converges at all points on the unit circle $|z| = 1$ except possibly $z = 1$.

Exercise 6 **(a)** *Explain why Rudin's Theorem 3.44 gives the convergence above for Boas' alternating harmonic power series.*

(b) *Give an example of a series satisfying the hypotheses of Rudin's Theorem 3.44 which also converges at $z = 1$.*

2 Values

Inside the unit disk $B_1(0) = \{z \in \mathbb{C} : |z| < 1\}$, we can write define $f : B_1(0) \to \mathbb{C}$ by

$$f(z) = \sum_{j=0}^{\infty}(-1)^j \frac{z^{j+1}}{j+1}.$$

This expression, as we have seen, is also well-defined when $|z| = 1$ and $z \neq -1$. In general, we can write

$$f(z) \sim \sum_{j=0}^{\infty} (-1)^j \frac{z^{j+1}}{j+1}$$

even when z takes values for which the series diverges. This is called **formal power series representation** but should not be confused with actual representation of a function by a power series when the series diverges.

The sum of the alternating harmonic series $f(1)$ is $\ln(2) \approx 0.69315$:

Exercise 7 *Find the power series expansion of* $\ln(1 + x)$ *with center* $x = 0$. *Then plug in* $x = 1$.

As suggested by Breandon Yeats, we can also compute numerically the values around the circle (at least to get some idea).

Exercise 8 *Show that whenever* $f(z)$ *is defined by a convergent series*

$$f(\overline{z}) = \overline{f(z)}.$$

As a consequence we need only consider points with non-negative argument between 0 and π. Figure 1 gives a plot of computed points corresponding to

$$\sum_{j=0}^{k} (-1)^j \frac{z^{j+1}}{j+1} \qquad \text{for } z = m\pi/10,\ m = 0, 1, 2, \ldots, 9 \text{ and } k = 100 \text{ and } k = 1000.$$

In Figure 2 we have added one additional point computed with $z = 999\pi/1000$. It is purple and was computed with $k = 100000$.

Many of the computations gave accuracy complaints, but I'm guessing the approximations are actually not too bad. In particular, it is probably correct that the values of f along the unit circle form a convex curve resembling a shifted inverse secant curve, namely $x = -\sec y + 1 + \ln 2$. It is noteworthy that the asymptotic limiting value of the imaginary part of $f(z)$ as $\mathrm{Arg}(z)$ tends to π appears to tend to $\pi/2$. I have no idea how to prove these assertions. We have plotted our calculated points along with this curve in Figure 3.

It is virtually certain that by "continuity" the real part of $f(z)$ tends to the limiting value of $f(-1) = -\infty$ as $z = e^{i\theta}$ tends to $e^{i\pi} = -1$. I haven't proved this either, but maybe you can.

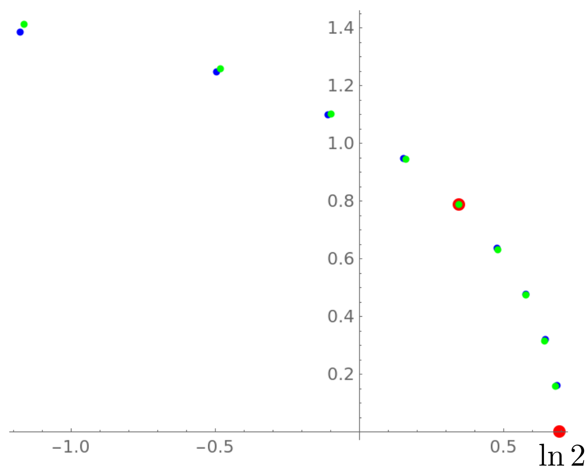

Figure 1: The points for $f(0)$ and $f(e^{i\pi/4})$ are computed with 100,001 terms of the series and are shown in red. The points in blue are computed with 101 terms of the series, and the points in green are computed with 1001 terms. As you can see, there is not much movement from the blue to the green to the red, so we probably have a reasonable numerical representation of the values.

Postscript

When I wrote above "I have no idea how to prove these assertions," it was true enough. After sleeping on it...I guess for a couple nights...it occured to me that perhaps I might know how to verify at least some of these assertions. In fact it is, more or less, obvious that one has an explicit formula for the function $f(z)$ and can answer just about any question concerning the values of Boas' alternating harmonic power series. The answer is so surprising, beautiful, and fundamental that I'm surprised it is not mentioned in, for example, baby Rudin. Maybe it is, but I haven't seen it.

It occurs to me that this is a very good opportunity for you to learn and appreciate several interesting aspects of complex analysis. So I'm not going to tell you the "obvious" observation which tells you everything, but I'll leave you with a couple hints and Figure 4. Hints: Think carefully about Exercise 7 above, Chapter 2 Sections 6 and 7 of Boas, and Problem 10 of Assignment 1.

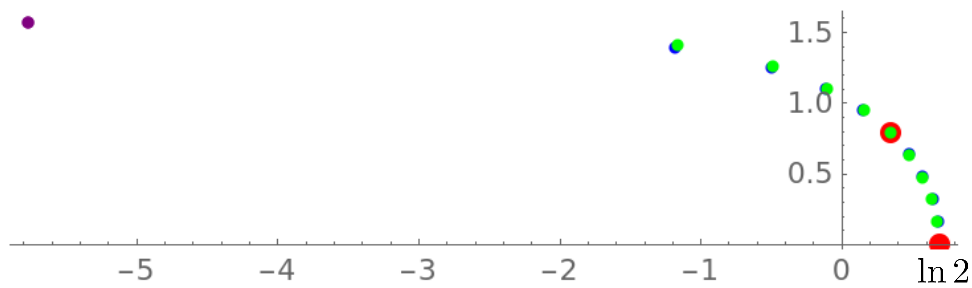

Figure 2: Inclusion of a point corresponding to $f(999\pi/1000)$.

Notes

The most informative compilation, related to this problem, we found on the internet was a post at `https://mathoverflow.net/questions/49395/behaviour-of-power-series-on-t` It was here I was reminded of the discussion in "baby" Rudin (which is what mathematicians call Rudin's *Principles of Mathematical Analysis*). One also finds there that the problem has come to be phrased as "What kind of subsets of the circle $\partial B_1(0)$ can be realized as the convergence set for a complex power series $\sum a_j z^j$ (with radius of convergence $r = 1$)?"

The following, for example, are known:

> *Given any closed subset A of the circle, there exists a complex power series for which A is precisely the set of points where the series converges.* (Mazurkiewicz)

> *Given any set A which is an F_σ set in the circle, there exists a complex power series for which A is precisely the set of points where the series converges.* (Herzog and Piranian)

An F_σ set is one which is a **countable union of closed sets**. Such a set has the form

$$A = \bigcup_{j=1}^{\infty} A_j \qquad \text{where } A_j \text{ is a closed set for } j = 1, 2, 3, \ldots.$$

I invite you to consider how complicated such a set can be.

> *There is a G_δ set which is* **not** *a set of convergence.* (T.W. Körner)

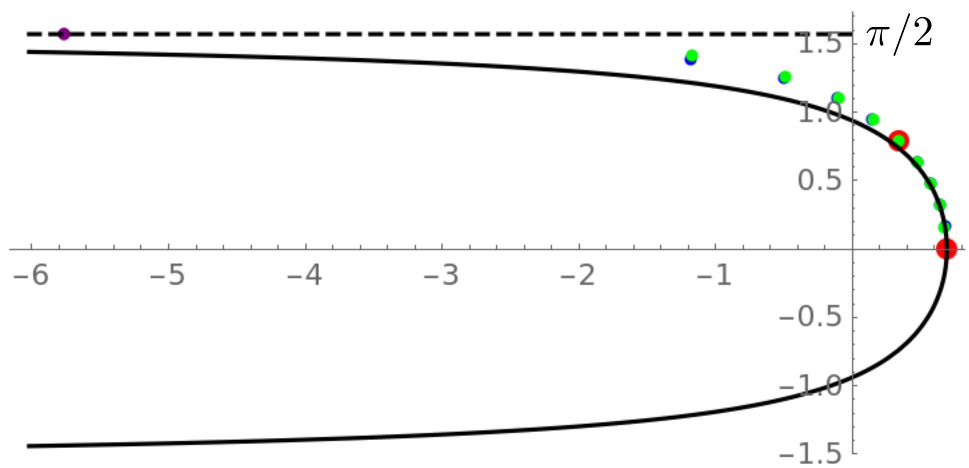

Figure 3: Image points along with $\mathrm{Re}(z) = 1 + \ln 2 - \sec \mathrm{Im}(z)$.

A G_δ is a bf countable intersection of open sets:

$$A = \bigcap_{j=1}^{\infty} U_j \qquad \text{where } U_j \text{ is an open set for } j = 1, 2, 3, \ldots.$$

And here is an open question:

> *Is every $G_{\delta\sigma}$ set of Lebesgue measure zero precisely the set of* **divergence** *points of some complex power series.*

As you may have guessed, a $G_{\delta\sigma}$ is a countable union of countable intersections of open sets:

$$\bigcup_{j=1}^{\infty} \left(\bigcap_{k=1}^{\infty} U_{jk} \right).$$

Many interesting mathematicians have thought about this kind of problem including Ted Kazcynski, the unibomber (who wrote his thesis at Berkeley under one of the authors, Piranian and Herzog, of the seminal papers on the subject), Paul Erdős (see the 1993 movie *N is a Number*), Yitzhak Katznelson (who wrote the wonderful and famous book *An Introduction to Harmonic Analysis*), and T.W. Körner (who wrote the book *Fourier Analyis* and has been called the best mathematical expositor of all time).

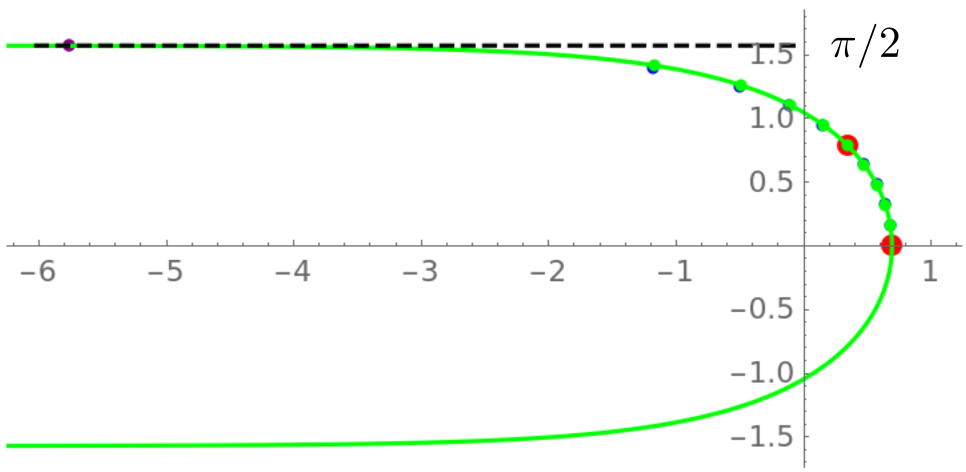

Figure 4: Image points along with the actual image curve determined by $f(z)$.