Assignment 2
Due Friday September 4, 2020

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Problem 1 *(Boas 2.7.2) Discuss the domain of convergence of the complex alternating harmonic power series:*

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

Problem 2 *(Boas 2.10.3) Find all the fourth roots of* 1*. These are also called the* **fourth roots of unity**.

Problem 3 *Find the real and imaginary parts of the following complex values and compare what you get to what a mathematical software package will tell you.*

(a) *(Boas 2.14.2)* $\log(-i)$.

(b) *(Boas 2.14.8)* $i^{2/3}$.

(c) *(Boas 2.14.11)* 2^i.

Problem 4 *Find a linear fractional transformation determined by the following values:*

$$\begin{array}{rcr} 1 & \mapsto & -1 \\ -1 & \mapsto & 1 \\ 0 & \mapsto & i. \end{array}$$

What is the image of ∞ *under your transformation?*

Problem 5 *Show that the complex tangent function has a complex period of* π*. This means*

$$\tan(z+\pi) = \tan z \qquad \textit{for every } z \in \mathbb{C}.$$

Problem 6 *Draw a picture illustrating how it is possible for the function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$ to be conformal at all points in $\mathbb{C}$ except $0 \in \mathbb{C}$.*

Problem 7 **(a)** *(Boas 14.2.30) Compute the complex derivative of $f(z) = z^n$ for n a positive integer.*

(b) *(Boas 14.2.33) Use the series definition of the complex exponential and termwise differentiation (from Chapters 1 and 2 of Boas if you want to read up on it) to compute the derivative of $f(z) = e^z$.*

The remaining problems rely on an identification between points $(x, y) \in \mathbb{R}^2$ and $z = x + iy \in \mathbb{C}$. According to this identification, we may refer to an open subset $\mathcal{U}$ of $\mathbb{C}$ and the corresponding open set in $\mathbb{R}^2$ by the same name $\mathcal{U}$. We can write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

for

$$f : \mathcal{U} \to \mathbb{C} \quad (\mathcal{U} \subset \mathbb{C}) \qquad \text{and} \qquad u : \mathcal{U} \to \mathbb{R} \quad (\mathcal{U} \subset \mathbb{R}^2).$$

Sometimes we will take $\mathcal{U} = \mathbb{C}$ (and $\mathcal{U} = \mathbb{R}^2$ by identification).

Problem 8 *Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function given by*

$$L(x, y) = (x + y, y)$$

and $f : \mathbb{C} \to \mathbb{C}$ be the corresponding complex function given by $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ where $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ by

$$u(x, y) = x + y \qquad \text{and} \qquad v(x, y) = y.$$

(a) *Show that L is infinitely (real) differentiable, i.e., L has partial derivatives in all components with respect to all variables of all orders.*

(b) *Show that f is not (complex) differentiable.*

Problem 9 *Let $\mathcal{U}$ be an open set as described above. Let $f : \mathcal{U} \to \mathbb{C}$ with $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be differentiable.*

(a) *(See page 669 in Boas) Show*

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}.$$

(b) *(See Theorem I on page 669 of Boas) Show*

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}. \end{cases}$$

These are called the **Cauchy-Riemann Equations**. *They constitute a system of partial differential equations.*

(c) *(See Theorem IV part 1 on page 672 of Boas and Problem 14.2.44 of Boas) Show*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

The (second order partial differential) operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is called the **Laplacian operator**, *and the partial differential equations $\Delta u = 0$ and $\Delta v = 0$ are (each) called* **Laplace's equation**.

Problem 10 **(a)** *Find a continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ with no first partial derivatives existing at one point.*

(b) *Find a continuous function $u : \mathbb{R}^2 \to \mathbb{R}$ with with both first partial derivatives continuous on $\mathbb{R}^2$ but no second partials existing at one point.*

(c) *Find a continuous function $j : \mathbb{R}^2 \to \mathbb{R}$ with both first partial derivatives and all three (or four) second partials existing and continuous but no third partials existing at one point.*