Assignment 3
Due Friday September 25, 2020

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September 22, 2020

Problem 1 **(a)** *Look up and write down* **all** *the properties that define a* **vector space** *over a field.*

(b) *Verify in detail that each of the properties defining a vector space holds for the vector space*

$$\mathbb{C}^2 = \{(z, w) : z, w \in \mathbb{C}\}$$

over the field $\mathbb{C}$ of complex numbers.

Problem 2 *(Boas 3.7.1) If $\mathbf{w} \in \mathbb{R}^n$ is fixed, is $f : \mathbb{R}^n \to \mathbb{R}$ by*

$$f(\mathbf{v}) = \mathbf{v} \cdot \mathbf{w} + 3$$

linear? If f is linear, express the values of f using matrix multiplication.

Problem 3 *(Boas 3.7.5) If $\mathbf{w} \in \mathbb{R}^3$ is fixed, is $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by*

$$\mathbf{F}(\mathbf{v}) = \mathbf{v} \times \mathbf{w}$$

linear? If $\mathbf{F}$ is linear, express the values of $\mathbf{F}$ using matrix multiplication.

Problem 4 *Let $C^0[a, b]$ denote the set of real valued continuous functions defined on the interval $[a, b]$.*

(a) *Show that $C^0[a, b]$ is a (real) vector space.*

(b) *Consider* $f : C^0[a,b] \to \mathbb{R}$ *by*

$$f[\phi] = \int_a^b \phi(x)\, dx.$$

Is the function f *linear? Can you express* f *using matrix multiplication?*

(c) *Consider the function* $g : C^0[a,b] \to \mathbb{R}$ *by*

$$g[\phi] = \phi\left(\frac{a+b}{2}\right).$$

Is the function g *linear?*

(d) *Consider the function* $\mathbf{F} : C^0[a,b] \to C^0[a,b]$ *by*

$$\mathbf{F}[\phi] = \phi^2.$$

Is $\mathbf{F}$ *linear?*

Problem 5 *(Boas 3.7.13) Let* $C^2(a,b)$ *denote the set of* **twice continuously differentiable** *real valued functions on the interval* (a,b)*.*

(a) *Show that* $C^2(a,b)$ *is a (real) vector space.*

(b) *Consider* $f : C^2(a,b) \to C^0(a,b)$ *by*

$$f[\phi] = x^2 \frac{d^2\phi}{dx^2} + (1+x^2)\frac{d\phi}{dx} + \phi.$$

That is, $f[\phi]$ *is the function* $\psi : (a,b) \to \mathbb{R}$ *given by*

$$\psi(x) = x^2 \frac{d^2\phi}{dx^2}(x) + (1+x^2)\frac{d\phi}{dx}(x) + \phi(x).$$

Is the function f *linear?*

Problem 6 *(Boas 3.7.16) Let* $M_n(\mathbb{C})$ *denote the set of all* $n \times n$ *matrices with complex entries.*

(a) *Show that* $M_n(\mathbb{C})$ *is a (complex) vector space.*

(b) *Consider $f : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by*

$$f(A) = A^{-1} \qquad \textit{(the inverse matrix of } A\textit{).}$$

Is the function f linear? If so, express f using matrix multiplication.

Problem 7 *(Boas 3.7.28) Consider $L : \mathbb{R}^3 \to \mathbb{R}^3$ where $L\mathbf{v}$ is the right-handed (counterclockwise) rotation of $\mathbf{v}$ about the x-axis. Express L using matrix multiplication (with respect to the standard basis).*

Problem 8 *(polynomials) Let*

$$Q[z] = \{a_2 z^2 + a_1 z + a_0 : a_2, a_1, a_0 \in \mathbb{C}\}$$

be the collection of all quadratic (and lower degree) polynomials in a complex variable z.

(a) *Show $Q[z]$ is a vector space over $\mathbb{C}$.*

(b) *Let $L : Q[z] \to Q[z]$ by $L[p] = p'$ (the complex derivative). Show that L is linear.*

(c) *Find a basis for $Q[z]$ and express L using matrix multiplication with respect to the basis you have chosen.*

Problem 9 *(Boas 3.7.22) Consider the transformation of the plane given by $T(x, y) = (\xi, \eta)$ where*

$$\xi = \frac{1}{\sqrt{2}}(x + y) \qquad \text{and} \qquad \eta = \frac{1}{\sqrt{2}}(-x + y)$$

(a) *Find $T(1, 0)$, $T(0, 1)$, and $T(1, 1)$. Draw a picture to indicate the action of the function f using these values.*

(b) *Describe the transformation T of the plane in (five) words.*

Problem 10 *Consider the nonlinear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$f(x, y) = (\sin(3x), y^2).$$

(a) *Can you draw a picture indicating the action of the function f on the plane?*

(b) *Find a linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ which approximates f to first order near the point $(x_0, y_0) = (0, 0)$.*