Assignment 4
Due Monday October 2, 2020

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Problem 1 *Describe all linear functions* $L : \mathbb{C} \to \mathbb{C}$.

Solution: $L(z) = L(1)z$. This means there is a fixed complex number $L(1) \in \mathbb{C}$ and L is given simply by multiplication by this number.

It may (and should) be noted as well that $L(1)$ can be written in the form

$$L(1) = re^{i\theta}$$

for some real numbers $r > 0$ and $\theta \in [0, 2\pi)$. Similarly,

$$z = |z|e^{i \operatorname{Arg} z}.$$

Thus,

$$L(z) = r|z|e^{i(\operatorname{Arg} z + \theta)}.$$

In this way we see L is a rotation (by θ) of the plane followed by a dilation or (isotropic) radial scaling (by $r > 0$). These (rotation and scaling) commute, so they can be executed in either order.

Finally, then one should observe that while dilation followed by scaling always gives a linear transformation of $\mathbb{R}^2$, there are many other linear transformations of $\mathbb{R}^2$. Thus, the collection of complex linear functions $L : \mathbb{C} \to \mathbb{C}$ can be compared to the collection of linear transformations $L : \mathbb{R}^2 \to \mathbb{R}^2$, and in that comparison is seen to be a strikingly **much smaller** collection. For example, the projection onto the real part $L : \mathbb{C} \to \mathbb{C}$ by $L(z) = \operatorname{Re} z$ is not linear. Do you see why? (Hint: Take a look at $L(ai)$ where a is real. This should be $iL(a)$. What is the projection onto the real part of ia?)

Problem 2 *Say* $L : \mathbb{R}^2 \to \mathbb{R}^1$ *is a linear function with one-dimensional kernel spanned by a vector* $\mathbf{v} \neq \mathbf{0}$. *Let* $\mathbf{w} \in \mathbb{R}^2 \setminus \ker L$.

(a) *Show that every vector* $\mathbf{x} \in \mathbb{R}^2$ *can be written uniquely as*

$$\mathbf{x} = a\mathbf{w} + b\mathbf{v},$$

i.e., $\{\mathbf{v}, \mathbf{w}\}$ *is a basis for* $\mathbb{R}^2$.

(b) *If* $\mathbf{x} \in \mathbb{R}^2$ *and*

$$\mathbf{x} = a\mathbf{w} + b\mathbf{v} = c\mathbf{v}^\perp + d\mathbf{v}$$

where $\mathbf{v}^\perp$ *is the (counterclockwise) rotation by* $\pi/2$ *of* $\mathbf{v}$, *then what is the relation between* a *and* b *and* c *and* d*?*

(c) *If* L *is expressed in coordinates using the basis* $\{\mathbf{v}^\perp, \mathbf{v}\}$ *for the domain* $\mathbb{R}^2$ *and the basis* $\{L\mathbf{w}\}$ *for the codomain, then what is the matrix of* L*?*

Solution: The question here revolves around the systems

$$\begin{cases} (\mathbf{v} \cdot \mathbf{w})\, a & + & (\mathbf{v} \cdot \mathbf{v})\, b & = & \mathbf{x} \cdot \mathbf{v} \\ (\mathbf{w} \cdot \mathbf{w})\, a & + & (\mathbf{v} \cdot \mathbf{w})\, b & = & \mathbf{x} \cdot \mathbf{w} \end{cases} \tag{1}$$

and

$$\begin{cases} & |\mathbf{v}|^2 d & = & \mathbf{x} \cdot \mathbf{v} \\ |\mathbf{v}|^2 c & & = & \mathbf{x} \cdot \mathbf{v}^\perp \end{cases} \tag{2}$$

for the coefficients. For part (a), we know the system has a unique solution given by the Cramer's rule formula (that's just a straightforward calculation) as long as the determinant of the coefficient matrix is nonzero. That is, the desired existence and uniqueness follows if we can show

$$(\mathbf{v} \cdot \mathbf{w})^2 - |\mathbf{v}|^2 |\mathbf{w}|^2 \neq 0. \tag{3}$$

In fact, the Cauchy-Schwarz inequality says that $\mathbf{v} \cdot \mathbf{w} \leq |\mathbf{v}||\mathbf{w}|$, so the quantity in (3) will be strictly negative unless equality holds in the Cauchy-Schwarz inequality. One proof of the Cauchy-Schwarz inequality goes like this: For any real number t

$$|\mathbf{w} + t\mathbf{v}|^2 \geq 0$$

with equality only if $\mathbf{w} + t\mathbf{v} = 0$ (because, for example, any inner product induced norm is positive definite). On the one hand, if we had $\mathbf{w} = -t\mathbf{v}$, then we would have

$L\mathbf{w} = -tL\mathbf{v} = 0$. This would mean $\mathbf{w} \in \ker(L)$, so that gives a contradiction. Thus, we can start with a strict inequality

$$|\mathbf{w} + t\mathbf{v}|^2 > 0$$

On the other hand, expanding the square of the norm in terms of the inner product (which in this case is the Euclidean dot product) we get

$$0 < |\mathbf{w}|^2 + 2(\mathbf{v} \cdot \mathbf{w})t + t^2|\mathbf{v}|^2.$$

We know here that $|\mathbf{v}|^2 > 0$, so this quadratic polynomial in t achieves its minimum value as a function of t when $t = -(\mathbf{v} \cdot \mathbf{w})/|\mathbf{v}|^2$. That is, we can take $t = -(\mathbf{v} \cdot \mathbf{w})/|\mathbf{v}|^2$ and conclude

$$0 < |\mathbf{w}|^2 - 2\frac{(\mathbf{v} \cdot \mathbf{w})^2}{|\mathbf{v}|^2} + \frac{(\mathbf{v} \cdot \mathbf{w})^2}{|\mathbf{v}|^2}.$$

Rearranging this inequality gives $(\mathbf{v} \cdot \mathbf{w})^2 < |\mathbf{v}|^2|\mathbf{w}|^2$ which implies (3), so we are done.

An alternative argument for part (a) may be given using standard coordinates as follows: Instead of (1), in standard coordinates we obtain the system

$$\left\{ \begin{array}{ccccc} w_1 a & + & v_1 b & = & x_1 \\ w_2 a & + & v_2 b & = & x_2 \end{array} \right. \tag{4}$$

where $w_1 = \mathbf{w} \cdot \mathbf{e}_1$, $w_2 = \mathbf{w} \cdot \mathbf{e}_2$, $v_1 = \mathbf{v} \cdot \mathbf{e}_1$, and so on. We know that this system has a unique solution (given by Cramer's rule) if $w_1 v_2 - v_1 w_2 \neq 0$. Remember, as mentioned above, we know $\mathbf{w}$ is not a multiple of $\mathbf{v}$. We need to show this implies the non-vanishing of the quantity

$$w_1 v_2 - v_1 w_2$$

which I guess can be done, but is kind of a pain in the neck. Let's see. If $w_1 v_2 = v_1 w_2$, then in the case $v_2 \neq 0$, we can solve for w_1 to find

$$\mathbf{w} = (v_1 w_2 / v_2, w_2) = (w_2/v_2)\mathbf{v}.$$

Since that is a contradiction, we must have $v_2 = 0$ and $\mathbf{v} = v_1 \mathbf{e}_1$. Also, we must have (assuming $w_1 v_2 = v_1 w_2$) that

$$v_1 w_2 = 0.$$

This means $w_2 = 0$, since $\mathbf{v} = v_1 \mathbf{e}_1 \neq 0$. Therefore, $\mathbf{w} = w_1 \mathbf{e}_1$. Again, $\mathbf{w} \in \ker(L)$, and we get a contradiction. I guess there were only two cases to consider.

In any case, the implication of these contradictions is that $w_1v_2 - v_1w_2 \neq 0$ as we were supposed to show.

This brings us to part (b). Notice that the second system (2) has solution

$$c = \frac{\mathbf{x}\cdot\mathbf{v}^\perp}{|\mathbf{v}|^2}, \quad d = \frac{\mathbf{x}\cdot\mathbf{v}}{|\mathbf{v}|^2}.$$

Substituting $\mathbf{x} = a\mathbf{w} + b\mathbf{v}$ in particular, we get

$$c = a\frac{\mathbf{w}\cdot\mathbf{v}^\perp}{|\mathbf{v}|^2}, \quad d = a\frac{\mathbf{w}\cdot\mathbf{v}}{|\mathbf{v}|^2} + b.$$

That is, there is a **linear relation**

$$\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \mathbf{w}\cdot\mathbf{v}^\perp/|\mathbf{v}|^2 & 0 \\ \mathbf{w}\cdot\mathbf{v}/|\mathbf{v}|^2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

This relation can be written, alternatively, as

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{|\mathbf{v}|^2}{\mathbf{w}\cdot\mathbf{v}^\perp}\begin{pmatrix} 1 & 0 \\ -\mathbf{w}\cdot\mathbf{v}/|\mathbf{v}|^2 & \mathbf{w}\cdot\mathbf{v}^\perp/|\mathbf{v}|^2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}.$$

For this second expression, we need to note that $\mathbf{w}\cdot\mathbf{v}^\perp \neq 0$. Do you see why? (Hint: If you substitute $\mathbf{x} = \mathbf{w}$ in the system for the coefficients c and d (and assume $\mathbf{w}\cdot\mathbf{v}^\perp = 0$) then you find $\mathbf{w} = d\mathbf{v}$.)

These relations correspond to a linear function $q : \mathbb{R}^2 \to \mathbb{R}^2$ and its inverse.

If you were given the matrix of L with respect to the basis $\mathcal{B} = \{\mathbf{w}, \mathbf{v}\}$ (in the domain) and you wanted to find the matrix for L with respect to the basis $\mathcal{N} = \{\mathbf{v}^\perp, \mathbf{v}\}$ (for the domain of L), then this linear relation would be helpful for you. I didn't make working out the details of this part of the problem, because I didn't want to make it too long. But you can work out those details in your spare time.

Instead I gave you part (c): Find the matrix of L using $\mathcal{N}$ for the domain and $\{L\mathbf{w}\}$ for the co-domain of L. This means, we need to find the images of the basis vectors $\mathbf{v}^\perp$ and $\mathbf{v}$ and write those as the columns of the matrix. The image of $\mathbf{v}^\perp$ is, well, $L\mathbf{v}^\perp$, but that is not very helpful since we need to express this vector in terms of the basis $\{L\mathbf{w}\}$. Let's use the system (1) of part (a) to write $\mathbf{v}^\perp$ as

$$\mathbf{v}^\perp = \frac{|\mathbf{v}|^2\mathbf{w}\cdot\mathbf{v}^\perp}{|\mathbf{v}|^2|\mathbf{w}|^2 - (\mathbf{v}\cdot\mathbf{w})}\mathbf{w} - \frac{(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v}^\perp)}{|\mathbf{v}|^2|\mathbf{w}|^2 - (\mathbf{v}\cdot\mathbf{w})}\mathbf{v}.$$

Therefore,

$$L\mathbf{v}^{\perp} = \frac{|\mathbf{v}|^2\mathbf{w}\cdot\mathbf{v}^{\perp}}{|\mathbf{v}|^2|\mathbf{w}|^2 - (\mathbf{v}\cdot\mathbf{w})} L\mathbf{w},$$

and that's our first column. Naturally, $L\mathbf{v} = 0$, so the matrix is the row vector

$$\left(\frac{|\mathbf{v}|^2\mathbf{w}\cdot\mathbf{v}^{\perp}}{|\mathbf{v}|^2|\mathbf{w}|^2 - (\mathbf{v}\cdot\mathbf{w})},\ 0\right).$$

Now, you could change basis to $\mathcal{B}$ for the domain and use the matrix for the change of basis q to see how the matrix for L changes.