Assignment 4
Due Monday October 2, 2020

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September 25, 2020

Problem 1 *Describe all linear functions* $L : \mathbb{C} \to \mathbb{C}$.

Problem 2 *Say* $L : \mathbb{R}^2 \to \mathbb{R}^1$ *is a linear function with one-dimensional kernel spanned by a vector* $\mathbf{v} \neq \mathbf{0}$. *Let* $\mathbf{w} \in \mathbb{R}^2 \setminus \ker L$.

(a) *Show that every vector* $\mathbf{x} \in \mathbb{R}^2$ *can be written uniquely as*

$$\mathbf{x} = a\mathbf{w} + b\mathbf{v},$$

i.e., $\{\mathbf{v}, \mathbf{w}\}$ *is a basis for* $\mathbb{R}^2$.

(b) *If* $\mathbf{x} \in \mathbb{R}^2$ *and*

$$\mathbf{x} = a\mathbf{w} + b\mathbf{v} = c\mathbf{v}^\perp + d\mathbf{v}$$

where $\mathbf{v}^\perp$ *is the (counterclockwise) rotation by* $\pi/2$ *of* $\mathbf{v}$, *then what is the relation between* a *and* b *and* c *and* d?

(c) *If* L *is expressed in coordinates using the basis* $\{\mathbf{v}^\perp, \mathbf{v}\}$ *for the domain* $\mathbb{R}^2$ *and the basis* $\{L\mathbf{w}\}$ *for the codomain, then what is the matrix of* L?

Problem 3 *(Boas 3.14.4) Consider the vector space*

$$P_2[x] = \{a_3x^3 + a_2x^2 + a_1x + a_0 : a_j \in \mathbb{R}, j = 0, 1, 2, \dots, 3\}$$

(over $\mathbb{R}$*).*

(a) *Find a linear function* $L_0 : P_3[x] \to \mathbb{R}$ *for which*

$$\ker L_0 = \{a_3x^3 + a_2x^2 + a_1x : a_j \in \mathbb{R}, j = 1, 2, \ldots, 3\}$$

Extra credit for giving a formula for $L_0[p]$ *which does not involve the coefficient expression for* p *directly. Hint: Use evaluation of* p*.*

(b) *Consider* $L_1; P_3[x] \to \mathbb{R}$ *by* $L_1[p] = p''(0)$*. What is* $\ker L_1$*?*

(c) *If* $L_2 : P_3[x] \to \mathbb{R}$ *has*

$$\dim\{L_2[p] : p \in P_3[x]\} = 2,$$

what is the dimension of the kernel of L_2*?*

Problem 4 *Let* $\alpha_1, \alpha_2 \in \mathbb{R}\backslash\{0\}$*, and consider* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_1 x \\ \alpha_2 y \end{pmatrix}.$$

(a) *Find the eigenvalues and eigenvectors of* L*.*

(b) *Show that*

$$\left\{L\begin{pmatrix} x \\ y \end{pmatrix} : x^2 + y^2 = 1\right\}$$

is an ellipse with semiaxes having lengths $|\alpha_1|$ *and* $|\alpha_2|$*.*

(c) *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be counterclockwise rotation of* $\mathbb{R}^2$ *by the angle* $\theta = \pi/4$*. Find the matrix* A *such that*

$$T \circ L\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}.$$

(d) *Find real numbers* a*,* b*, and* c *such that*

$$\left\{T \circ L\begin{pmatrix} x \\ y \end{pmatrix} : x^2 + y^2 = 1\right\} = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : ax^2 + bxy + cy^2 = 1\right\}$$

Problem 5 *(General Fact about linear functions) If* $L : V \to W$ *is linear and one-to-one, then* $\ker(L) = \{0_V\}$*.*

Problem 6 *(linear automorphisms of a finite dimensional space) Consider a linear function $L : \mathbb{R}^n \to \mathbb{R}^n$. Show the following:*

(a) *If L is one-to-one, then L is onto.*

(b) *If L is onto, then L is one-to-one.*

Problem 7 *Consider*

$$\frac{d}{dx} : P[x] \to P[x]$$

where $P[x]$ denotes the vector space of polynomials with real coefficients (over $\mathbb{R}$). Show that d/dx is onto but not one-to-one.

Problem 8 *Find an example of a linear function $L : V \to V$ which is one-to-one but not onto.*

Problem 9 *(products and geometry in $\mathbb{R}^3$) The* **cross product** *of two vectors*

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

is defined by

$$\mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ -(x_1 y_3 - x_3 y_1) \\ x_1 y_2 - x_2 y_1 \end{pmatrix}.$$

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be **parallel** *if $\mathbf{x} \times \mathbf{y} = \mathbf{0}$.*

The **dot product** *of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ is defined by*

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{3} x_j y_j.$$

Two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be **perpendicular** *if $\mathbf{x} \cdot \mathbf{y} = 0$. Show the following conditions hold whenever $\mathbf{x}, \mathbf{y} \in \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are standard basis vectors. Then use linearity to obtain the general assertion.*

(a) $\mathbf{y} \times \mathbf{x} = -\mathbf{x} \times \mathbf{y}$.

(b) $(\mathbf{x} \times \mathbf{y}) \cdot (\mathbf{z} \times \mathbf{w}) = (\mathbf{x} \cdot \mathbf{z})(\mathbf{y} \cdot \mathbf{w}) - (\mathbf{x} \cdot \mathbf{w})(\mathbf{y} \cdot \mathbf{z})$.

(c) $(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{y} \cdot \mathbf{z})\mathbf{x}$.

(d) $\mathbf{x} \times \mathbf{y}$ *is perpendicular to* $\mathbf{x}$ *and* $\mathbf{y}$.

Problem 10 *(Boas 3.8.5-6) A plane through the origin in* $\mathbb{R}^3$ *with normal* $\mathbf{n}$ *is given by*

$$P = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} \cdot \mathbf{n} = 0\}.$$

Let $\mathbf{v}, \mathbf{w} \in P$ *be two non-parallel vectors.*

(a) *If* $\mathbf{x}$ *is any vector in* P*, show* $\mathbf{x}$ *can be written uniquely as a linear combination*

$$\mathbf{x} = a\mathbf{v} + b\mathbf{w} \tag{1}$$

of $\mathbf{v}$ *and* $\mathbf{w}$*. Hint: Take the cross product of both sides of (1) with* $\mathbf{v}$ *and then with* $\mathbf{w}$ *to solve for the coefficients* a *and* b*.*

(b) *Look up and write down* **Cramer's rule** *for the solution of two linear equations in two unknowns* x_1 *and* x_2*:*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

(c) *Show that the coefficients* a *and* b *obtained in part (a) using the hint can be written as quotients of* 2×2 *determinants as in Cramer's rule. Hint: Take the dot products of both sides of (1) with* $\mathbf{v}$ *and* $\mathbf{w}$*.*