Assignment 5
Due Monday October 23, 2020

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October 24, 2020

Problem 1 *In Problem 5 of Exam 2 the linear function* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 2y \\ 2x - y \end{pmatrix}$$

was considered, and in part (b)(v) you were asked to show that the points in the image $L(\partial B_1(\mathbf{0})) = \{L(x,y)^T : x^2 + y^2 = 1\}$ *satisfy an equation*

$$5x^2 - 4xy + 8y^2 = 36.$$

Consider a change of variables determined by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \xi \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} + \eta \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}.$$

(a) *Draw a picture illustrating this change of variables. (Boas' version of this picture is her Figure 11.2 of Chapter 3 Section 11 on page 151.)*

(b) *Write down the quadratic equation in the variables* ξ *and* η *with the coefficient of the middle* ($\xi\eta$) *term in the form*

$$A\sin(2\phi) + B\cos(2\phi).$$

You'll need to use the double angle formulas

$$\begin{cases} \cos(2\phi) &= \cos^2\phi - \sin^2\phi \\ \sin(2\phi) &= 2\cos\phi\sin\phi. \end{cases}$$

(c) *Find the unique angle* $\phi \in (-\pi/4, \pi/4)$ *for which the coefficient of* $\xi\eta$ *vanishes.*

(d) *Given your choice of* ϕ *from part (b), find the values of* $\cos(2\phi)$ *and* $\sin(2\phi)$*.*

(e) *Express the coefficients of* ξ^2 *and* η^2 *in terms of* $\cos(2\phi)$ *and* $\sin(2\phi)$*. You will probably want to use the identities*

$$\begin{cases} \cos^2\phi &= \frac{1}{2}(\cos^2\phi + 1 - \sin^2\phi) = \frac{1}{2}(1+\cos(2\phi)) \\ \\ \sin^2\phi &= \frac{1}{2}(1-\cos(2\phi)). \end{cases}$$

(f) *Find an equation of the form*

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1$$

satisfied by the points in the image of the unit circle under L*. (Find the lengths of the semi-axes* a *and* b*.)*

(g) *Why are the lengths of the semi-axes not the same as those of the ellipse in part (iv) of part (b) of Problem 2 on Exam 2?*

Problem 2 *Consider the standard linear shear* $\Sigma : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$\Sigma\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

(a) *Draw a picture indicating the geometrical significance of the shear angle* θ*.*

(b) *Rotate coordinates as in the previous problem to find an equation of the form*

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1$$

satisfied by the points in the image of the unit circle under Σ*. (Find the lengths* a *and* b *in terms of* θ*.)*

Problem 3 *The following presentation of some of the material from Chapater 7 of Boas is borrowed from the famous text An Introduction to Harmonic Analysis by Yitzhak Katznelson. Let* $\mathcal{P}$ *denote the vector space of complex valued continuous periodic functions*

$$f : \mathbb{R} \to \mathbb{C}$$

with period 2π. *Let* $\mathcal{S}$ *denote the vector space of* **double ended sequences** *of complex numbers*

$$\{a_j\}_{j=-\infty}^{\infty} = \{\ldots a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, \ldots\}.$$

Each double ended sequence is also a function $a : \mathbb{Z} \to \mathbb{C}$ *where* $\mathbb{Z}$ *denotes the integers and* $a(j) = a_j$.

We will consider two important norms on $\mathcal{P}$. *The first is the* L^1 *norm given by*

$$\|f\|_{L^1} = \frac{1}{2\pi} \int_0^{2\pi} |f(t)|\, dt.$$

The other is the L^2 *norm given by*

$$\|f\|_{L^2} = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2\, dt \right)^{1/2}.$$

(a) *Show that*

$$\frac{1}{2\pi} \int_0^{2\pi} e^{ijt}\, dt = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}$$

(b) *Examples of functions in* $\mathcal{P}$ *are the* **trigonometric polynomials** $p : \mathbb{R} \to \mathbb{C}$ *by*

$$p(t) = \sum_{j=-k}^{k} a_j e^{ijt}.$$

The integers j *are called the* **frequencies** *of* p.

Show that the coefficients of a trigonometric polynomial satisfy (i.e., are given by)

$$a_j = \frac{1}{2\pi} \int_0^{2\pi} p(t)\, e^{-ijt}\, dt.$$

(c) *Consider the Fourier (coefficient) transform* $C : \mathcal{P} \to \mathcal{S}$ *by*

$$C(f) = \hat{f} \qquad \text{where} \qquad \hat{f}(j) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-ijt}\, dt.$$

(i) *Show* C *is linear.*

(ii) *Show the Fourier transform of the complex conjugate of* f *satisfies the following:*

$$C(\overline{f})(j) = \overline{\hat{f}(-j)}.$$

This may be written equivalently as

$$\widehat{\overline{f}}(j) = \overline{\hat{f}(-j)}.$$

(iii) *If* $T : \mathcal{P} \to \mathcal{P}$ *by* $T(f)(t) = f(t - \tau)$, *then (show)*

$$C(T(f))(j) = \hat{f}(j)e^{-ij\tau}.$$

Equivalently,

$$\widehat{T(f)}(j) = \hat{f}(j)e^{-ij\tau}.$$

The function T *is called a* **translation operator**.

(iv) *Show*

$$|\hat{f}(j)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(t)|\,dt = \|f\|_{L^1}.$$

Problem 4 *Consider the ordinary differential equation*

$$\frac{du}{dx} = 0 \tag{1}$$

for a real valued function u *defined on an interval* (a, b) *and having one continuous derivative. The vector space of continuously differentiable real valued functions is denoted by* $C^1(a, b)$.

(a) *Find all solutions of (1).*

(b) *How do you know you have all of them?*

(c) *Is the solution set*

$$\Sigma = \{u \in C^1(a, b) : u' = 0\}$$

a vector space? If so, what is the dimension?

Problem 5 *(Boas 8.1.1) Verify that*

$$u(x) = ae^{-x} + be^{x}$$

and

$$v(x) = a\cosh x + b\sinh x$$

are solutions of the ODE $u'' = u$ *for any coefficients* a *and* b.

Problem 6 *(Boas 8.1.2) Solve the two point boundary value problem*

$$\begin{cases} u'' = u \\ u(0) = 0 \\ u(\ln 2) = 3/4. \end{cases}$$

Hint: Use the solutions of the form $v(x) = a\cosh x + b\sinh x$.

Problem 7 *Consider the ODE* $y'(x) = f(x)$ *for the function* y *where* $f \in C^0(a, b)$.

(a) *Find all solutions of the equation in* $C^1(a, b)$. *Think carefully about the limits of integration.*

(b) *Is the solution set for this equation a subspace of* $C^1(a, b)$*?*

Problem 8 *Find a single second order equation satisfied by each unknown function in the following system of first order equations:*

$$\begin{cases} u' = v \\ v' = -u. \end{cases}$$

Problem 9 *Verify that* $u : \mathbb{R} \to \mathbb{C}$ *by* $u(t) = e^{i\omega t}$ *and* $v : \mathbb{R} \to \mathbb{C}$ *by* $v(t) = i\omega e^{i\omega t}$ *satisfy the system*

$$\begin{cases} u' = v \\ v' = -\omega^2 u. \end{cases}$$

Problem 10 *Given a positive real frequency* ω, *find all solutions* $\mathbf{u} = (u_1, u_2)^T : \mathbb{R} \to \mathbb{C}$ *of the system*

$$\mathbf{u}' = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \mathbf{u}.$$

Hint: Look at the two previous problems. Find another solution like the one in the previous problem and take a linear combination of those two.