Assignment 6
Due Friday October 30, 2020

John McCuan

October 24, 2020

Problem 1 *Recall that a function* $u : (a, b) \to \mathbb{R}$ *is* **differentiable** *or* **classically differentiable** *at* $x \in (a, b)$ *if the limit*

$$\lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \qquad \text{exists.}$$

If $u : (a, b) \to \mathbb{R}$ *is differentiable at every point* $x \in (a, b)$*, then we say* u *is differentiable on* (a, b) *and write* $u \in \mathcal{D}^1(a, b)$ *or* $u \in \text{Diff}(a, b)$*.*

A function $u : (a, b) \to \mathbb{R}$ *is* **continuous** *at* $x_0 \in (a, b)$ *if for any* $\epsilon > 0$*, there is some* $\delta > 0$ *such that the following holds:*

> *Whenever* $x \in (a, b)$ *and* $|x - x_0| < \delta$*, then* $|u(x) - u(x_0)| < \epsilon$*.*

Similarly to differentiability, we say $u : (a, b) \to \mathbb{R}$ *is continuous on* (a, b) *is* u *is continuous at every point and write* $u \in C^0(a, b)$*.*

Show a differentiable function is continuous.

Problem 2 **(a)** *Find a function which is in* $C^0(a, b)$ *but not differentiable (at all points of* (a, b)*).*

(b) *Find a function which is differentiable but not in* $C^1(a, b)$*.*

The set of (classically) differentiable functions may be denoted by $\mathcal{D}^1(a, b)$ *or* $\text{Diff}(a, b)$*, so*

$$C^1(a, b) \subsetneq \text{Diff}(a, b) \subsetneq C^0(a, b).$$

Problem 3 *(C^∞ and C^ω functions) The set of functions having k continuous derivatives is denoted by $C^k(a,b)$.*

$$C^\infty(a,b) = \bigcap_{k=1}^{\infty} C^k(a,b).$$

A function $u : (a,b) \to \mathbb{R}$ is said to be **real analytic** *if for any $x_0 \in (a,b)$ there is some $r > 0$ for which*

$$u(x) = \sum_{j=0}^{\infty} \frac{u^{(j)}(x_0)}{j!}(x-x_0)^j \qquad \text{for } |x-x_0| < r.$$

Note that this requries the series to converge and also that the values of the convergent series equal those of the function on $B_r(x_0) = \{x \in \mathbb{R} : |x-x_0| < r\}$. The set of real analytic functions is denoted by $C^\omega(a,b)$.

Most of the (infinitely) differentiable functions you know are real analytic.

Find a function in $C^\infty(\mathbb{R}) \backslash C^\omega(\mathbb{R})$.

Problem 4 *(support) For this problem you need some definitions:*

A subset U *of* $\mathbb{R}$ *is* **open** *if for each* $x \in U$*, there is some* $r > 0$ *with*

$$B_r(x) = \{\xi \in \mathbb{R} : |x - \xi| < r\} \subset U.$$

A subset $A \subset \mathbb{R}$ *is* **closed** *if the complement* $A^c = \mathbb{R} \backslash A$ *is open.*

(a) *Prove that every intersection of closed sets is closed. That is, if* Γ *is any (indexing) set and* A_α *is closed for* $\alpha \in \Gamma$*, then*

$$\bigcap_{\alpha \in \Gamma} A_\alpha \qquad \text{is closed.}$$

The **closure** *of any set* A *is the intersection of all closed sets containing* A*. We can denote the closure of a set* A *by* $\overline{A}$ *or* $\operatorname{clos}(A)$*. Thus,*

$$\overline{A} = \bigcap_{\substack{X \text{ closed} \\ A \subset X}} X.$$

(b) *Show that the closure is well-defined by showing* $\mathbb{R}$ *is closed. (Why does this show the closure is well-defined?)*

The **support** *of a function* $u : (a, b) \to \mathbb{R}$ *is the closure (in* $\mathbb{R}$*) of the set of points where* $u \neq 0$*. That is,*

$$\operatorname{supp}(u) = \overline{\{x \in (a, b) : u(x) \neq 0\}}.$$

(c) *Find the support of* $f : (0, \infty) \to \mathbb{R}$ *given by*

$$f(x) = \sin\left(\frac{1}{x}\right).$$

The set of all C^∞ **functions with compact support in** (a, b) *is denoted by*

$$C_c^\infty(a, b) = \{u \in C^\infty(a, b) : \operatorname{supp}(u) \subset (a, b)\}.$$

(d) *Show that if* $u \in C_c^\infty(0, 1)$*, then* $0 \notin \operatorname{supp}(u)$*. Hint:* $\operatorname{supp}(u)$ *is closed.*

(e) *Show that* $C_c^\infty(a, b)$ *is not empty.*

Problem 5 *(weakly differentiable functions) In Problem 1 above I explained to you very precisely the definitions of continuous and differentiable functions. Even though you may not have known those definitions before, you've been using those functions for years. For this problem we need two more function spaces which are a tad too complicated for me to explain to you in full detail. Nevertheless, you can understand these spaces intuitively and use them. I will, however, use proper notation for the concepts related to these function spaces. This will also be new to you but intuitively clear.*

The **integral** *of a function* $u : (a, b) \to \mathbb{R}$ **over a set** $A \subset (a, b)$ *is denoted by*

$$\int_A u.$$

If I wish to distinguish a variable of integration ξ*, I will do it like this:*

$$\int_{\xi \in A} u(\xi) \qquad \text{or} \qquad \int_{\xi \in A} u.$$

Compare these to the notation

$$\int_a^b u(\xi)\, d\xi \qquad \textit{used for continuous functions.}$$

Here is the main source of missing details: I will refer to certain sets as being **measurable***. These sets include every set you have ever encountered. You can (for the time being) think of the measurable sets as* **all sets** *(but of course they are not all sets or I wouldn't be talking about them).*

Our first function space is called $L^1_{loc}(a, b)$*. This is the space of* **locally integrable functions***:*

$$L^1_{loc}(a, b) = \left\{ u : \int_A |u| \textit{ is well-defined for every measurable set with } \overline{A} \subset (a, b) \right\}.$$

Any set A *with* $\overline{A} \subset (a, b)$ *is said to be* **compactly contained** *in* (a, b)*. In this case we write* $A \subset\subset (a, b)$*.*

The second space is $W^1(a, b)$*, the set of* **weakly differentiable functions***. A function* $u \in L^1_{loc}(a, b)$ *is* **weakly differentiable** *with weak derivative* $v \in L^1_{loc}(a, b)$ *if*

$$-\int_{(a,b)} u\phi' = \int_{(a,b)} v\phi \qquad \textit{for every } \phi \in C_c^\infty(a, b).$$

(a) *Show $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = |x|$ has $g \in W^1(a,b)$. What is the weak derivative of g?*

(b) *Show $C^1(a,b) \subset W^1(a,b)$. Hint: Integration by parts.*

Given $f \in L^1_{loc}(a,b)$, a function $u \in W^1(a,b)$ is said to be a **weak solution** *of the ordinary differential eqation*

$$u' = f$$

if the following condition holds:

$$-\int_{(a,b)} u\phi' = \int_{(a,b)} f\phi \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

(c) *Consider the Heaviside function $h : \mathbb{R} \to \mathbb{R}$ by*

$$h(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

Find a weak solution of the ODE $u' = h$.

Problem 6 *(Boas 6.3) Solve the following first order linear equations. Indicate an appropriate open interval (a,b) for which your solution satisfies $y \in C^1(a,b)$. Solve the IVP associated with your equation and $y(x_0) = y_0$ for $x_0 \in (a,b)$. Try to determine the interval (a,b) before/without solving the ODE.*

(a) $(1 + e^x)y' + 2e^x y = (1 + e^x)e^x$.

This equation is linear and non-singular on $\mathbb{R}$, so we expect a solution in $C^1(\mathbb{R})$.

First Solution: We write the equation in standard form

$$y' + \frac{2e^x}{1 + e^x}\, y = e^x.$$

The integrating factor is

$$\mu(x) = e^{\int^x 2e^t/(1+e^t)\, dt} = e^{2\ln(1+e^x)} = (1 + e^x)^2.$$

Thus, we write the equation as

$$[(1 + e^x)^2 y]' = (1 + e^x)^2 y' + 2e^x(1 + e^x)y = e^x(1 + e^x)^2 = e^{3x} + 2e^{2x} + e^x.$$

Integrating both sides from x_0 to x, we get

$$(1+e^x)^2 y - (1+e^{x_0})^2 y(x_0) = \frac{1}{3}e^{3x} + e^{2x} + e^x - \left(\frac{1}{3}e^{3x_0} + e^{2x_0} + e^{x_0}\right).$$

Therefore,

$$y(x) = \frac{1}{(1+e^x)^2}\left[(1+e^{x_0})^2 y(x_0) + \frac{1}{3}e^{3x} + e^{2x} + e^x - \left(\frac{1}{3}e^{3x_0} + e^{2x_0} + e^{x_0}\right)\right].$$

Second Solution: Proceed as in the first solution until the integration from x_0 to x. At that point just write

$$(1+e^x)^2 y = \frac{1}{3}e^{3x} + e^{2x} + e^x + c$$

Therefore,

$$y(x) = \frac{e^{3x} + 3c}{3(1+e^x)^2} + \frac{e^x}{1+e^x}.$$

If $y(x_0) = y_0$, then

$$c = (1+e^{x_0})^2 y_0 - \frac{1}{3}e^{3x_0} - e^{2x_0} - e^{x_0},$$

and this should give us the same solution we had above.

(b) $2xy' + y = 2x^{5/2}$.

(c) $y' + y = e^x$.

Problem 7 *The ODE $x' = \alpha x$ where α is a constant is very simple. It is autonomous, linear (and homogeneous). You should know it inside and out.*

(a) *Find the equilibrium points and plot the phase diagrams corresponding to the three obvious cases.*

(b) *Find the general solution in terms of the initial value $x(0) = x_0$, and plot solutions in solution space for the three cases.*

Problem 8 *Write the following system of first order equations in the form $\mathbf{x}' = A\mathbf{x}$ for some vector valued function $\mathbf{x}$ and some constant matrix A.*

$$\begin{cases} u' = v \\ v' = -u. \end{cases}$$

Problem 9 *Compute*

$$e^{tA} = \sum_{j=0}^{\infty} \frac{(tA)^j}{j!}$$

where A is your matrix from Problem 8. Hint: Compute the powers of A first, and express them in terms of the 2×2 identity matrix I and A. Then let t go along for the ride to find

$$e^{tA} = f(t)A + g(t)I$$

for some appropriate (familiar) real analytic functions f and g.

Problem 10 *Apply your computed matrix e^{tA} from Problem 9 to an initial value*

$$\mathbf{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

to verify that the general solution of the system of equations $\mathbf{x}' = A\mathbf{x}$ from Problem 8 is

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0.$$

(a) *Compare this to your general solution from Problem 7(b).*

(b) *Can you make a guess about the general solution of $\mathbf{x}' = A\mathbf{x}$ where A is any constant matrix?*

(c) *How would you describe the solution you have obtained here for a fixed initial point $\mathbf{x}(0) = \mathbf{x}_0$?*