Assignment 7
Due Friday November 13, 2020

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Problem 1 *In Problem 3 of Assignment 6, you were asked to find a function in* $C_c^\infty(a,b)$*. You might have found the following function which is sometimes referred to as the* standard bump function*:*

$$\phi(\mathbf{x}) = \begin{cases} e^{-1/(1-|\mathbf{x}|^2)}, & |\mathbf{x}| < 1 \\ 0, & |\mathbf{x}| \ge 1. \end{cases}$$

You may note that $\phi : \mathbb{R}^n \to \mathbb{R}$ *and* $\phi \in C_c^\infty(B_{1+\epsilon}(0))$ *for every* $\epsilon > 0$*. This includes* $\phi \in C_c^\infty(-1-\epsilon, 1+\epsilon)$ *for every* $\epsilon > 0$ *when* $n = 1$*. Also in the case* $n = 1$*, setting* $m = (a+b)/2$ *and* $r = (b-a)/2$*, the function* $\psi : \mathbb{R} \to \mathbb{R}$ *by* $\psi(x) = \phi(2(x-m)/r)$ *gives an example in* $C_c^\infty(a,b)$*.*

(a) *Make sure you understand the properties of* ϕ*. Use mathematical software to plot the graph of* ϕ*. Prove that all derivatives of* ϕ *are well-defined and vanish on the boundary of* $\operatorname{supp}(\phi)$*.*

(b) *Given* $\delta > 0$*, use* ϕ *to find an even nonnegative function* $\mu \in C_c^\infty(\mathbb{R})$ *with the following properties*

(i)

$$\int \mu = \int_{\mathbb{R}} \mu = \int_{-\infty}^{\infty} \mu(x)\, dx = 1.$$

(ii) $\operatorname{supp}(\mu) = [-\delta, \delta]$.

Of course, what you have found is actually a one-parameter family of functions depending on the parameter δ*. Nevertheless,* μ *is called the* **standard mollifier** *or an* **approximate identity***.*

(c) *Plot the graph of* μ *for* $\delta = 1, 1/10, 1/100$.

(d) *Find*

$$\lim_{\delta \searrow 0} \mu(0).$$

Problem 2 *Let* μ *be the standard mollifier and let* $u \in L^1_{loc}(a,b)$ *with* $-\infty < a < b < \infty$. *Given* $\delta > 0$, *the* **mollification** *of* u *is* $\mu * u : (a+\delta, b-\delta) \to \mathbb{R}$ *by*

$$\mu * u(x) = \int_{\xi \in (a,b)} \mu(x-\xi) u(\xi). \tag{1}$$

If you're having trouble with the notation in the integral above, note that in the case when $u \in C^0(a,b)$ *the integral can be written as*

$$\mu * u(x) = \int_a^b \mu(x-\xi) u(\xi)\, d\xi.$$

This is called a **convolution integral**. *The convolution integral, and the mollification in particular, has some remarkable properties.*

(a) *Show* $\mu * u \in C^\infty(a+\delta, b-\delta)$. *Hint: Look at where the* x *dependence is in the convolution integral. This will tell you what the derivative should be.*

(b) *Show*

$$\mu * u(x) = \int_{\xi \in (a,b)} \mu(\xi - x) u(\xi)$$

so that μ *in the integrand is a copy of* μ *"recentered" at* x. *Let* x *be fixed and draw a picture of the graph of* $\mu(\xi - x)$ *and the graph of a continuous function* $u \in C^0(a,b)$, *both as functions of* ξ, *on the same set of axes near the point* $x \in (a,b)$. *Hint: You do not know* $u(x)$, *but the values of* $u(\xi)$ *for* ξ *near* x *are close to* $u(x)$.

(c) *Show the convolution is commutative, i.e.,* $\mu * u = u * \mu$.

(d) *Given that* $u \in C^0(a,b)$, *find*

$$\lim_{\delta \searrow 0} \mu * u(x).$$

Hint: This is why μ *is called an approximate identity.*

Problem 3 *In Problem 5 of Assignment 6 you were supposed to observe that the weak formulation of an FTC equation $y' = f$ may be obtained as follows:*

(i) *Assume you have a classical solution $u \in C^1(a,b)$ and integrate both sides against a test function $\phi \in C_c^\infty(a,b)$ like this:*

$$\int_a^b u'(x)\phi(x)\,dx = \int_a^b f(x)\phi(x)\,dx.$$

(ii) *Integrate by parts on the left to obtain*

$$-\int_a^b u(x)\phi'(x)\,dx = \int_a^b f(x)\phi(x)\,dx.$$

(iii) *Notice that the condition*

$$-\int_{(a,b)} u\phi' = \int_{(a,b)} f\phi \qquad \text{for all } \phi \in C_c^\infty(a,b) \tag{2}$$

makes sense even when u and f are only in $L^1_{loc}(a,b)$ (and not even necessarily continuous).

(iv) *Finally, forget about the original form of the equation $y' = f$, and take (2) to be the* **definition** *of what it means to have a weak solution $u \in L^1_{loc}(a,b)$ of the ODE $u' = f$ with $f \in L^1_{loc}(a,b)$.*

Notice that the definition of a weak solution $u \in L^1_{loc}(a,b)$ of the FTC equation $u' = f \in L^1_{loc}(a,b)$ is also the definition of what it means for f to be the weak derivative of u. (Surprise, surprise!)

You also had the opportunity to show $g(x) = |x|$ is a weak solution of $u' = f$ where $f(x) = -h(-x) + h(x)$ and h is the standard Heaviside function. Let μ be the standard mollifier.

(a) *Determine explicitly and plot $\mu * g$.*

(b) *Determine explicitly and plot $\mu * f$ where $f(x) = -h(-x) + h(x)$ as above.*

(c) *Show*

$$(\mu * g)' = \mu * f \qquad \text{classically.}$$

(d) *Formulate a conjecture concerning the mollification of a weak solution* $u \in L^1_{loc}(a,b)$ *of the ODE* $u' = f \in L^1_{loc}(a,b)$. *Hint: What ODE does* $\mu * u$ *solve classically? Be careful about the domain of definition for the functions in your conjecture; remember that in parts (a) and (b) above we had* $g, h \in L^1_{loc}(\mathbb{R})$. *Can you prove your conjecture?*

Autonomous ODE

An ODE is **autonomous** if it has the form

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}).$$

There are two main "spaces" associated with an autonomous ODE: **phase space** and **solution space**. Usually phase space is of more interest; this is the Euclidean space $\mathbb{R}^n$ which may be taken as the co-domain of solutions $\mathbf{x} : (a, b) \to \mathbb{R}^n$. In the case $n = 1$ under consideration in the first part of our discussion of ODEs, we can write

$$x' = f(x)$$

and take phase space to be the **phase line** $\mathbb{R}^1$. To **draw a phase diagram** usually involves several steps and may involve a multitude of steps including plotting the images of solutions using mathematical software. The basic objective, however, is to **understand the orbit structure** of the autonomous equation. An **orbit** is a subset of phase space: The orbit of the point $\mathbf{x}_0$ determined by the ODE $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is

$$\mathcal{O}(\mathbf{x}_0) = \{\mathbf{x}(t) : t \in (a, b)\}$$

where (a, b) is the domain of $\mathbf{x}$ and $\mathbf{x}$ is the (hopefully unique) solution of the IVP

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0. \end{cases}$$

Notice it is assumed here that $t \in (a, b)$. An orbit is usually drawn in phase space by drawing orbits which are curves and putting arrows on them. The glaring exception is the plotting of equilibrium points, which are not curves but only points. The two steps always involved in drawing a phase diagram are (1) finding and plotting equilibrium points and (2) plotting representative orbits indicating the overall orbit structure. Other important steps may be the plotting of **null-clines** and **separatrices** as well as **ω-limit sets**. The following problems will lead you through plotting some phase diagrams.

Solution space for an ODE, on the other hand, is a cross product of the domain (an interval) of solutions $\mathbf{x}$ with the co-domain of $\mathbf{x}$. Solution space is, in principle, of interest for any ODE, but since the domain of different solutions of the same ODE may be different intervals (and for other reasons) solution space may be more complicated.

Problem 4 *An ODE is* **linear** *if it has the form* $L(\mathbf{x}) = \mathbf{v}(t)$ *where* L *is a* **linear ordinary differential operator** *and* $\mathbf{v}$ *is a given function of the independent variable* t *called the* **inhomogeneity** *or* **forcing function**. *If* $\mathbf{v} \equiv \mathbf{0}$, *then the linear ODE is said to be* **homogeneous**. *Associated with every linear ODE is, as you know, an associated homogeneous ODE.*

Linear ODEs are usually not autonomous. Under what conditions is the linear first order (single) ODE $Lx = f$ *autonomous? To be an* **autonomous first order ODE** *in this context means* $L : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ *depends only on* x *and* x'. *Thus, you may assume* L *can be written in the form*

$$Lx = G(x', x)$$

where $G : \mathbb{R}^2 \to \mathbb{R}$. *Hint: Show* G *is linear.*

Problem 5 *Assume* $u \in C^1(\mathbb{R})$ *by*

$$u(x) = \begin{cases} -x^2, & x \leq 0 \\ x^2, & x \geq 0 \end{cases}$$

is a solution of the autonomous ODE $y' = f(y)$.

(a) *Solve the IVP*

$$\begin{cases} y' = f(y) \\ y(0) = 5. \end{cases}$$

(b) *Plot all solutions of* $y' = f(y)$ *in solution space. (Be careful!)*

Problem 6 *Every autonomous equation is* **separable**, *and every separable equation can be solved (at least implicitly). It is often the case that the information one desires about an autonomous equation can be obtained without solving the equation, but this problem asks you to solve a well-know autonomous equation. The* **logistic equation** *used to model population dynamics is given by*

$$P' = \alpha P(K - P)$$

where K *is a particular population called the* **carrying capacity** *and* $\alpha K > 0$ *is a nominal growth constant.*

(a) *Divide both sides of the logistic equation by* $P(K - P)$ *and integrate from time* $t = 0$ *to time* t.

(b) *Change variables on the left using $\xi = P(t)$ and then integrate using partial fractions in ξ. (Be careful to change the limits of integration when you change variables.)*

(c) *Solve for $P = P(t)$, and calculate the limit*

$$\lim_{t\to\infty} P(t).$$

(d) *Draw the phase line diagram associated with the logistic equation and prove, using the existence and uniqueness theorem, that the value of the limit you have obtained is correct.*

In general, an ODE is said to be **separable** *if it has the form $g(x)x' = q(t)$.*

Problem 7 *In the special case in which all solutions of a given ODE are defined and unique for all real times $t \in \mathbb{R}$, the set of solutions is called a* **dynamical system**.

(a) *Show that if two orbits in a dynamical system intersect, then they are the same orbit.*

(b) *If $\mathcal{O}(\mathbf{x}_0) = \{\mathbf{x}(t) : \mathbf{x}' = \mathbf{F}(\mathbf{x})\}$ is an orbit of a dynamical system and there is some $t_0 \in \mathbb{R}$ and some $p > 0$ for which $\mathbf{x}(t_0) = \mathbf{x}(t_0 + p)$, then $\mathbf{x}$ is said to be* **periodic** *with period p and $\mathcal{O}(\mathbf{x}_0)$ is said to be a* **periodic orbit**. *Show that $\mathbf{x}(t) = \mathbf{x}(t + p)$ for every $t \in \mathbb{R}$ when $\mathcal{O}(\mathbf{x}_0)$ is a periodic orbit.*

(c) *Given a periodic orbit $\mathcal{O}(\mathbf{x}_0) = \{\mathbf{x}(t) : \mathbf{x}' = \mathbf{F}(\mathbf{x})\}$ of a dynamical system, show that if $\mathcal{O}(\mathbf{x}_0) = \{\mathbf{y}(t) : \mathbf{y}' = \mathbf{F}(\mathbf{y})\}$, then $\mathbf{x}$ and $\mathbf{y}$ have the same set of periods, i.e.,*

$$\{p \in (0, \infty) : \mathbf{x}(t + p) = \mathbf{x}(t)\} = \{p \in (0, \infty) : \mathbf{y}(t + p) = \mathbf{y}(t)\}.$$

Thus, it makes sense to talk about the period of an orbit.

(c) *Given a periodic orbit $\mathcal{O}(\mathbf{x}_0) = \{\mathbf{x}(t) : \mathbf{x}' = \mathbf{F}(\mathbf{x})\}$ of a dynamical system, consider*

$$p_0 = \inf\{p \in (0, \infty) : p \text{ is a period of } \mathcal{O}(\mathbf{x}_0)\}.$$

Show that if $p_0 > 0$, then p_0 is a period for $\mathcal{O}(\mathbf{x}_0)$. Hint: Take a solution generating the orbit, and use continuity.

(d) *What can you say if $p_0 = 0$?*

Problem 8 *Can there be a periodic orbit of a one-dimensional dynamical system, i.e., a dynamical system associated with a single autonomous ODE, with a positive period? Why or why not?*

Problem 9 *(Boas 2.16, 7.5.1, and 8.2.20) Resistors, capacitors, and inductors (coils) are electronic components with which are associated values of resistance (R), capacitance (C), and inductance (L) respectively. Charge q is assumed to move at a uniform rate through a circuit, and this rate is called current:*

$$I = \frac{dq}{dt}.$$

The component rules relating voltage (potential difference) and charge are the following

(i) *The voltage drop across a resistor of value R is*

$$V_R = IR$$

where I is the current in the circuit.

(ii) *The voltage drop across a capacitor of value C is proportional to the charge (difference) accumulated on the plates*

$$q = CV_C.$$

In particular, if you know the voltage across a capacitor (as a function of time), you can determine the current in the circuit:

$$I = C\frac{dV_C}{dt}.$$

(iii) *The voltage drop across a coil of inductance L is given by*

$$V_L = L\frac{dI}{dt}.$$

(a) *A circuit consists of a resistor, a capacitor, and a switch (in series in a loop). At time $t = 0$, with accumulated charge q_0 on the capacitor, the switch is thrown closed. Write down and solve an ODE for the charge accumulated on the capacitor as a function of time. Hint: You can use equation (1.2) on page 391 of Boas with supply voltage $V \equiv 0$ (after the switch is thrown) and $L = 0$.*

(b) *A circuit consists of a resistor, a coil attached end to end. At time $t = 0$ a magnet makes a one-time pass through the coil inducing an initial current I_0 in the circuit. Write down and solve an ODE for the current in the circuit as a function of time.*

(c) *A resistor, capacitor, and coil are in a circuit with a power supply. Equation (1.2) then becomes a second order linear equation for the charge:*

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V. \tag{3}$$

(i) *Find a particular solution of equation (3) of the form* $q(t) = q_0$ *(constant) when* $V \equiv v_0$ *is a constant.*

(ii) *Find a particular solution of equation (3) of the form* $q(t) = a\cos(jt) + b\sin(jt)$ *when* $V(t) = \sin(jt)$.

(d) *Say we want to find a particular solution of equation (3) when*

$$V(t) = \sum_{j=-\infty}^{\infty} h(t-(2j-1)\pi) - h(t-2j\pi)$$

where h *is the standard heaviside function. Note that there is no classical solution of this ODE in* $C^2(\mathbb{R})$. *The function* $g(t) = V(t) - 1/2$, *however, is an odd* 2π *periodic function. Therefore, it is possible to write*

$$V(t) = \frac{1}{2} + \sum_{j=1}^{\infty} v_j \sin(jt) \tag{4}$$

and use part (c) to find a series solution of the form

$$q(t) = q_0 + \sum_{j=1}^{\infty} [a_j \cos(jt) + b_j \sin(jt)]. \tag{5}$$

(i) *Plot* $V(t)$ *given in terms of the Heaviside function.*

(ii) *Integrate both sides of (4) from* $t = -\pi$ *to* $t = \pi$ *to find the coefficients* v_j. *Hint: Remember Problem 10 on Exam 2.*

(iii) *Use mathematical software to plot the first few terms of the Fourier series for* V.

(iv) *Use part (c) to determine the coefficients* q_0, a_j *and* b_j *for* $j = 1, 2, 3, \ldots$. *Hint: Pair up the terms in the series (4) with particular solutions in the series (5).*

(v) *Plot the first few terms in the series for your particular (weak) solution.*

Problem 10 *Consider the coupled system of logistic equations*

$$\begin{cases} x' = x(1 - x + y/2) \\ y' = y(5/2 - 3y/2 + x/4). \end{cases}$$

(a) *Writing* $\mathbf{x} = (x, y)^T$ *so that this system has the form* $\mathbf{x}' = \mathbf{F}(\mathbf{x})$*, identify the vector function* $\mathbf{F} = (f_1, f_2)^T$.

(b) *Find the equilibrium points for this system.*

(c) *Evaluate the matrix*

$$D\mathbf{F} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}$$

at each equilibrium point.

(d) *Use mathematical software to plot the vector field* $\mathbf{F}$ *near each equilibrium point* $\mathbf{x}_*$ *and the vector field given by*

$$\mathbf{v}(\mathbf{x}) = D\mathbf{F}(\mathbf{x}_*)\mathbf{x}.$$