Lecture 2: Complex Functions

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1 Review of Complex Numbers

This isn't really a full review, but I'm going to look back over **Chapter 2 of Boas** and see if there are any important topics I missed. You can take this as an opportunity to do the same. Perhaps a good goal for you is to be able to understand everything in Chapter 2 and be able to work any problem there. Everything we didn't cover should be relatively easy for you to pick up now.

The most significant and interesting omissions are the discussions of (complex) **power series** in § 6 and some material on **powers and roots** in § 10 and § 14. I've included the first five problems of Assignment 2 for you to consider whatever we might have glossed over in these sections.

On the other hand, we have talked about Riemann surfaces and conformal mapping, which are generally considered much more advanced topics. It might be useful to mention/emphasize once again that **multiplication by i** is **counterclockwise rotation by $\pi/2$**. That is, the function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = iz$$

corresponds to a counterclockwise rotation of $\mathbb{C}$ by an angle $\pi/2$. For example,

$$f(1) = i \qquad \text{and} \qquad f(i) = -1.$$

This simple geometric observation is central to what makes complex analysis very different from real analysis. This forces a certain rigidity, expecially when it comes to **complex differentiable functions** which is the topic we are about to address as we focus on Chapter 14 of Boas.

More generally, every complex number has a **polar form** $z = re^{i\theta}$ where r is the modulus and θ is the argument. The multiplication $zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$

may also be expressed as

$$\left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

We see then that complex multiplication corresponds to a multiplication of the moduli and a summing of the arguments. Of course, rotation is a kind of adding of arguments, so you should see how this is manifest in multiplication by i.

The Complex Domain and $\mathbb{R}^2$

As a final note before moving to the material of Chapter 14, we should not forget what we know (if anything) about real valued functions, and it might be worthwhile to take some note of how the new things we have learned relate to the "old" framework. We could, in principle, express the multiplication of complex numbers in terms of a weird multiplication of "points" or "vectors" in $\mathbb{R}^2$:

$$(a, b) * (c, d) = (ac - bd, ad + bc).$$

Hopefully, it's clear to you by now how cumbersome this would be and you appreciate, at least to some extent, the power and elegance of the complex notation and framework.

It is still useful, sometimes, to consider the identification between $\mathbb{R}^2$ and $\mathbb{C}$ given by

$$x + iy \longleftrightarrow (x, y).$$

We are also about to use this identification with functions, so this is a good time to point out some elementary aspects of what is involved. A complex function has the form

$$f(z) = f_1(z) + i f_2(z)$$

where f_1 and f_2 are real valued functions of a complex variable $z = x + iy$. We will rarely mention the real and imaginary parts of f using this notation, but rather we will think of f_1 and f_2 in terms of (more or less identical) real valued functions on (some subset of) $\mathbb{R}^2$:

$$u(x, y) = f_1(z) \qquad \text{and} \qquad v(x, y) = f_2(z).$$

If we call the domain of these functions $\mathcal{U} \subset \mathbb{R}^2$, then using our general notation for functions

$$u : \mathcal{U} \to \mathbb{R} \qquad \text{and} \qquad v : \mathcal{U} \to \mathbb{R}.$$

Exercise 1 *Write down carefully (using set construction notation and the domain $\mathcal{U}$) the complex domain $\mathcal{R} \subset \mathbb{C}$ associated with $\mathcal{U}$ so that*

$$f_1 : \mathcal{R} \to \mathbb{R} \qquad \text{and} \qquad f_2 : \mathcal{R} \to \mathbb{R}.$$

You have probably studied functions like u and v before and have at least a passing familiarity with graphs, level curves, partial derivatives, directional derivatives, and gradients associated with such functions. This might be a good time to review what you know about these things. You would have seen these things either in high school calculus or (in more detail) in Calculus III.

Exercise 2 *Let* $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ *by* $f(z) = 1/z$. *Find the real and imaginary parts of* f.

Exercise 3 *Let* $v : \mathbb{R}^2 \to \mathbb{R}$ *by* $v(x, y) = x^2 + y^2$.

(a) *Draw the graph and level curves of* v.

(b) *Calculate the directional derivative of* v *in the direction* $(1/2, \sqrt{3}/2)$ *at the point* $(1, 0) \in \mathbb{R}^2$.

Exercise 4 *Let* $u : \mathbb{R}^2\backslash\{(0, 0)\} \to \mathbb{R}$ *by* $u(x, y) = x/(x^2 + y^2)$.

(a) *Calculate all first and second partial derivatives of* u *and find the value of*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

(b) *Draw the level curves of* u *and illustrate how the gradient vector at a point* $(x, y) \in \mathbb{R}^2\backslash\{(0, 0)\}$ *is orthogonal to the level curve passing through this point.*

2 Differentiability and Conformality

When you look at generalizations of the "nice" real valued functions you know[1] (like polynomials, exponentials, and trigonometric functions) you generally get complex functions $f : \mathbb{C} \to \mathbb{C}$ which are both **differentiable** and **conformal**. It's useful to know what these terms mean and that they are, more or less, **equivalent**.

Differentiability at $z_0 \in \mathbb{C}$ for the function $f : \mathcal{U} \to \mathbb{C}$ means the limit

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad \text{exists.} \tag{1}$$

[1] Here, we're talking about functions $f : (a, b) \to \mathbb{R}$ which are **real valued** and defined on some open interval. For example, associated with $f(x) = x^2$, there is $f(z) = z^2$, associated with the real tangent function $\tan x$, there is the complex tangent function $\tan z$, and so on.

This looks rather a lot like the definition of differentiability for functions of one real variable in Calculus I. As in that case, the limiting quantity is called a **difference quotient**, and it is also customary to write the value of the limit (the complex derivative) as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

But a complex function for which this limit exists is very different from the differentiable functions you've seen before. Before we start to discuss this, you may be wondering about the symbol $\mathcal{U}$ used for the domain of the function f above. This is a little bit of a technicality, but let's address it before we go further.

Open Sets

Throughout this section, and generally when one discusses complex function theory, one considers complex valued functions $f : \mathcal{U} \to \mathbb{C}$ with domains $\mathcal{U} \subset \mathbb{C}$. A preliminary point is that $\mathcal{U}$ need not be all of $\mathbb{C}$. (More generally, $\mathcal{U}$ may be taken as a subset of a Riemann surface, but still, it need not be the entire surface.) We almost always want the domain of such a function to be **open**. This is a simple concept which you probably don't understand precisely. Here is the precise meaning:

For each $z_0 \in \mathcal{U}$, there is some $r > 0$ such that

$$\{z \in \mathbb{C} : |z - z_0| < r\} \subset \mathcal{U}. \tag{2}$$

The set on the left in (2) is called an **open ball** with center z_0 and radius r. The requirement is that every point $z_0 \in \mathcal{U}$ has some **space around it** where the value of the function f still makes sense. We do not want z_0 to be an **edge point** of $\mathcal{U}$, so what we are saying is that **none of the points in $\mathcal{U}$ is an edge point of $\mathcal{U}$**.

Exercise 5 *Show that $\{z \in \mathbb{C} : |z| \leq 1\}$ is* **not** *an open subset of $\mathbb{C}$.*

From now one, $\mathcal{U}$ will represent an open subset of $\mathbb{C}$.

This is, perhaps a reasonable place to offer one more "frame of reference" comment. As we consider complex differentiable functions $f : \mathcal{U} \to \mathbb{C}$, there are primarily three kinds of real functions about which you may know and which may be considered for comparison. These are

1. Real valued functions of a real variable, $f : (a, b) \to \mathbb{R}$. (Calculus I)

2. Real valued functions of two real variables, $u : \mathcal{U} \to \mathbb{R}$ where $\mathcal{U} \subset \mathbb{R}^2$. (Calculus III)

3. Vector valued functions of two real variables or **vector fields**, $\mathbf{v} : \mathcal{U} \to \mathbb{R}^2$ where $\mathcal{U} \subset \mathbb{R}^2$. (These were also considered to some extent in Calculus III and some special cases were considered in linear algebra. A linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by matrix multiplication is a special case of this kind of function.)

Exercise 6 *Of the three kinds of functions enumerated above, one of them is equivalent to a complex valued function of a complex variable $f : \mathcal{U} \to \mathbb{C}$ where $\mathcal{U} \subset \mathbb{C}$. Which one?*

Exercise 7 *Assuming you were able to successfully complete the previous exercise, take that kind of function and write down what it would mean for it to be* **differentiable** *with respect to its identification as a complex function.*

2.1 Conformality

There are various ways to illustrate what (1) implies about a function $f : \mathcal{U} \to \mathbb{C}$. **Conformality** means that **angles are preserved**. For this description to be meaningful, you need to know what are **angles** and what it might mean for angles to be **preserved**.

First of all, the simplest angles are those formed by two rays in standard position. That is, take the positive real axis $\{x : x \geq 0\}$ and some other ray

$$\ell = \{t(\cos\theta + i\sin\theta : t \geq 0\}.$$

If $0 \leq \theta \leq \pi$, then we say the **angle** between the positive real axis and ℓ is θ.

Exercise 8 **(a)** *If $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z + 1$, then draw the image of the positive real axis and the image of ℓ,*

$$f(\ell) = \{f(z) : z \in \ell\},$$

under the function f.

(b) *If $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$, then draw the image of the positive real axis and the image of ℓ,*

$$f(\ell) = \{f(z) : z \in \ell\},$$

under the function f.

The images of all the rays in the exercise were again rays, so it is not difficult to say what happens to the angles after applying a function f (in these cases). If the angle is the same, we say it is **preserved** under f. If it is different, then we say it is **not preserved**. Now you should have some idea of what it means for a complex function $f : \mathcal{U} \to \mathbb{C}$ to be **conformal**, i.e., to preserve angles.

Of course, not every complex function maps rays to rays. Generally, an image like

$$f(\ell) = \{f(z) : z \in \ell\} = \{f(t\cos\theta + it\sin\theta) : t \geq 0\}$$

will be some **curve** (and not on a straight line). But the angle between two curves (when they cross) at a point should make sense.

Exercise 9 **(a)** $\gamma(t) = (\cos t, \sin t)$ *parameterizes a circle, and so does* $\alpha(t) = (1-\cos t, \sin t)$. *Find where these curves intersect in the first quadrant and find the angle between them there.*

(b) *Repeat the first part of this exercise for the curves* $\gamma(t) = \cos t + i\sin t$ *and* $\alpha(t) = 1 - \cos t + i\sin t$.

If you were able to successfully complete Exercise 9, then you know finding angle between curves involves taking derivatives. These derivatives, furthermore, are different from the kinds of derivatives we've mentioned above. They are derivatives of a different kind of function, namely, a **vector valued function of one real variable** or, alternatively, a **complex valued function of one real variable**. The derivative of such a function, at least in the real case you may remember, is often called a tangent vector. For example, if $\mathbf{r}(t) = (2\cos t, 3\sin t)$, then the **image curve** has tangent vector $br'(t) = (-2\sin t, 3\cos t)$ at the point $\mathbf{r}(t)$.

Exercise 10 *Draw a careful picture of the image curve determined by the function*

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^2 \qquad \text{by} \qquad \mathbf{r}(t) = (2\cos t, 3\sin t)$$

and draw $\mathbf{r}'(\pi/4)$ *as a vector at the point* $\mathbf{r}(\pi/4)$.

Let's try to use what we've been discussing to see if the complex function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$ is conformal. You may remember that I suggested above that when you use a formula you know from one variable calculus, like $f(x) = x^2$, to define a complex valued function of a complex variable, then you usually get a conformal mapping.

If you've done part (b) of Exercise 8 above, then you should be skeptical about this assertion. If you do not understand the point I am trying to make, go back and do Exercise 8.

Let's take a different angle, namely, the positive real axis starting at $a > 0$ and the ray

$$\ell + a = \{z + a : z \in \ell\} = \{a + t\cos\theta + ti\sin\theta : 0 \le t < \infty\}.$$

These two rays form an angle of size θ at $a \in \mathbb{R}$. The positive real axis maps under $f(z) = z^2$ to the positive real axis as before. But the image of $\ell + a$ under $f(z) = z^2$ is not a ray:

$$(a + t\cos\theta + i\sin\theta)^2 = a^2 + 2at\cos\theta + t^2(\cos^2\theta - \sin^2\theta) + 2i(a + t\cos\theta)t\sin\theta.$$

Exercise 11 *Plot this parameterized curve using mathematical software.*

Notice that the image of the vertex $a \in \mathbb{R} \cap (\ell + a)$ is $a^2 \in \mathbb{R}^2$. To find the angle formed between the image of $\ell + a$ and the positive real axis, we should take a derivative:

$$\frac{d}{dt}\left[a^2 + 2at\cos\theta + t^2\cos(2\theta) + i[2at\sin\theta + t^2\sin(2\theta)]\right].$$

And then we should evaluate at $t = 0$. The value of the derivative is

$$2a\cos\theta + 2t\cos(2\theta) + i[2a\sin\theta + 2t\sin(2\theta)].$$

Evaluating at $t = 0$, we get

$$2a(\cos\theta + i\sin\theta).$$

This means the angle is preserved. In fact, all angles are preserved by $f(z) = z^2$ **except** angles at the origin. $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by $f(z) = z^2$ **is** conformal. The problem is that the derivative vanishes at $z = 0$.

Exercise 12 *Compute the complex derivative of $f(z) = z^2$ by finding the limit*

$$\lim_{z\to z_0} \frac{z^2 - z_0^2}{z - z_0}.$$

Here is the[2] theorem:

Theorem 1 *If $f : \mathcal{U} \to \mathbb{C}$ is differentiable on the open set $\mathcal{U}$ with $f'(z) \neq 0$ for $z \in \mathcal{U}$, then f is conformal on $\mathcal{U}$. Conversely, if $f : \mathcal{U} \to \mathbb{C}$ is conformal, then f is differentiable.*

This is a first indication that complex differentiability is much more restrictive than real differentiability.

[2] I'm not sure Boas states this result, but it's an important one.

Exercise 13 *Find a function* $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ *which is smooth (real differentiable) but does not preserve angles, i.e., find functions* $u : \mathbb{R}^2 \to \mathbb{R}$ *and* $v : \mathbb{R}^2 \to \mathbb{R}$ *such that* u *and* v *have real partial derivatives but the mapping*

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{by} \qquad f(x, y) = (u(x, y), v(x, y))$$

is not conformal. Hint: Try a linear function.

I have probably, more or less, covered enough material to prepare you to do Assignment 2. In the rest of these notes, I will briefly discuss some other results showing how special complex differentiable functions are. These are all basic results to know.

2.2 Regularity

First of all, you should know that real differentiable functions can be differentiable once but not twice. Consider $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \int_0^x |t|\, dt. \tag{3}$$

It is easy to see that $f'(x) = |x|$ but $f''(0)$ does not exist.

Exercise 14 **(a)** *Calculate the integral in (3) to obtain an explicit expression for* f *given using cases.*

(b) *Plot the function* f *defined in (3), it's derivative and its second derivative (where it is defined).*

(c) *Construct a function which has two derivatives at all points, but fails to have a third derivative at one point.*

Exercise 15 *Find a function* $u : \mathbb{R}^2 \to \mathbb{R}$ *with first partial derivatives but no second partial derivatives (at a point).*

Theorem 2 *If* $f : \mathcal{U} \to \mathbb{C}$ *is a complex differentiable function, then* f *is infinitely differentiable, i.e., the derivative* f' *of* f *is a complex differentiable function; the derivative* f'' *fo* f' *is a complex differentiable function, and so one.*

2.3 Analyticity

Here is another difference:

Theorem 3 *(see Theorem III on page 671 of Boas) If $\mathcal{U}$ is an open subset of $\mathbb{C}$ and $f : \mathcal{U} \to \mathbb{C}$ is complex differentiable at each point of $\mathcal{U}$, then f is infinitely differentiable at each point of $\mathcal{U}$, and given $z_0 \in \mathcal{U}$, there exists some $r > 0$ for which the series*

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j$$

converges for $|z - z_0| < r$ and satisfies

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!} (z - z_0)^j, \qquad |z - z_0| < r. \tag{4}$$

Functions which are represeted by power series are called **analytic**. Because of this (and Theorem 3) complex differentiable functions are often called **complex analytic functions**. Boas does this.

Exercise 16 *Find a function $f : \mathbb{R} \to \mathbb{R}$ such that f has derivatives of all orders at $x = 0$ and the series*

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j \qquad \text{converges for all } x \in \mathbb{R},$$

but

$$f(x) \neq \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j \qquad \text{for all } x > 0.$$

A real valued function which is represented by its power series is called **real analytic**. Analyticity is also a kind of regularity. In the study of functions of a real variable a function which is **continuous** is said to be in C^0. Put another way, the **set of all continuous functions** is denoted by C^0. Very often one denotes the domain of the function by writing $C^0(\mathcal{U})$ for the set of (real valued) continuous functions with domain $\mathcal{U}$. If one wishes to include the co-domain in this notation, one can use

$$C^0(\mathcal{U} \to \mathbb{R}^m) = \{f : f \text{ is a continuous function from } \mathcal{U} \text{ to } \mathbb{R}^m\}.$$

Exercise 17 *Write out what it means for f to be in $C^0(\mathcal{U} \to \mathbb{R}^m)$ in terms of the real valued coordinate functions $f_1, f_2, \dots, f_m$ of f and $C^1(\mathcal{U})$.*

The set of **continuously differentiable functions**, i.e., those functions with continuous first derivatives is denoted by C^1. The inclusion of domains and co-domains is just the same as for C^0, thus a real valued function defined on $\mathcal{U} \subset \mathbb{R}^n$ with continuous first order partial derivatives is said to be in $C^1(\mathcal{U})$. For real valued functions, having continuous first partial derivatives implies continuity. Therefore, $C^1 \subset C^0$.

You see how this is going to go now: A function with k continuous (real partial) derivatives is in C^k.

A function with (continuous) derivatives of **all** orders is said to be in C^∞.

A function which is represented by its power series (a real analytic function) is in C^ω.

$$C^\omega \subset C^\infty \subset \cdots \subset C^{k+1} \subset C^k \subset \cdots \subset C^1 \subset C^0.$$

Note that these sets are all meaningful for real valued functions: None of these sets are equal. Were we to extend this notation to complex valued functions $f : \mathcal{U} \to \mathbb{C}$ with $\mathcal{U} \subset \mathbb{C}$, then we would have

$$C^\omega(\mathbb{C}) = C^\infty(\mathbb{C}) = \cdots = C^{k+1}(\mathbb{C}) = C^k(\mathbb{C}) = \cdots = C^1(\mathbb{C}) \subset C^0(\mathbb{C}).$$

As a consequence, this notation is only used for real (vector) valued functions of one (or several) real variables.

2.4 Relations to PDEs

Theorem 4 *(See Theorem II on page 670 of Boas) If $u : \mathcal{U} \to \mathbb{R}$ and $v : \mathcal{U} \to \mathbb{R}$ with $\mathcal{U}$ considered as an open subset of $\mathbb{R}^2$ satisfy the Cauchy-Riemann Equations*

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}, \end{cases}$$

then $f : \mathcal{U} \to \mathbb{C}$ with $\mathcal{U}$ considered as an open subset of $\mathbb{C}$ is analytic.

Theorem 5 *(See Theorem IV on page 672 of Boas) If $\mathcal{U}$ is a* **simply connected** *open subset of $\mathbb{R}^2$ and $u : \mathcal{U} \to \mathbb{R}$ is* **harmonic** *in $\mathcal{U}$, i.e., u satisfies* **Laplace's equation**

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

then there exists a function $v : \mathcal{U} \to \mathbb{R}$ *for which*

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

is an analytic function.

Corollary 1 *(See Theorem III and Theorem IV of Boas) If* $u : \mathcal{U} \to \mathbb{R}$ *is harmonic (and twice differentiable), then* u *has partial derivatives of all orders. In this case we say* u *is* **infinitely differentiable** *and write*

$$u \in C^\infty(\mathcal{U}).$$

In fact, u *is real analytic. The collection of all real analytic functions on* $\mathcal{U}$ *is denoted by* $C^\omega(\mathcal{U})$.

2.5 Comments/Summary/Outline

It would be very nice to give full proofs of the results above (and many other beautiful results in complex analysis), but my aim is only to give kind of a superficial introduction to the subject. If you come away with a solid ability to do some complex arithmetic and an awareness of the statements of some of the basic results in complex function theory, then that is probably enough for now. I've picked out a few aspects of the main results which only involve an elementary understanding of the complex arithmetic and put them in the Assignment 2 problems and on Exam 1. Generally speaking, I'm very happy if you are interested in any and all aspects of the proofs, but it's probably more appropriate if you make time to take a dedicated course on complex analysis (later). In this course, we want to save (the majority of) the time for reviewing linear algebra and ODEs. In addition, there will be many opportunities for you to hone your skills at mathematical proof in those areas, which are probably more familiar to you.

Having said all that I would recommend that, if you are interested in any particular topic or proof that I'm not planning to cover, pick up a book and try to read something about it. There is also a tremendous amount of information available on the internet. Most of it is even correct. (There are even errors in math books sometimes—believe it or not!) Of course, also ask me about it, and I may be able to push you off in the right direction. It's likely I won't have full and correct proofs (and all the details) of complicated results right at the tips of my fingertips. (Only mediocre people are always at their best.)

Finally, here is an outline giving, off the top of my head, what I think you should know *a little bit* about concerning complex differentation after taking this course.

1. Definition of complex differentiation
2. Cauchy-Riemann equations
3. Laplace's equation
4. Regularity
5. Conformality
6. Analyticity, i.e., power series representation

That's probably enough.

3 Complex Integration

Hopefully you've noticed the difference between a complex valued function of a real variable $\gamma : (a,b) \to \mathbb{C}$ and a complex function of a complex variable $f : \mathcal{U} \to \mathbb{C}$. Both are used in complex integration. The image of a complex valued function of one real variable is generically **a curve** Γ, and it is over this curve that complex integration takes place. In symbols

$$\Gamma = \{\gamma(t) \in \mathbb{C} : t \in (a,b)\},$$

and we want to understand **the integral of a function** $f : \mathcal{U} \to \mathbb{C}$ **over** the curve Γ which we write as

$$\int_\Gamma f.$$

Of course, this assume $\Gamma \subset \mathcal{U}$, i.e., the curve lies in the domain of f.

There are various reasons to study complex integrals like this. One is that some of the "big theorems" stated above concerning differentiation are proved using complex integration, though we won't really get into the proofs. Another reason is that some (maybe many) real integrals whose values are difficult to compute in closed form can be computed using complex integrals. Roughly speaking such computations are based on **the calculus of residues**. I'm not sure how far we'll get into residue calculus, but you should be in a position to learn about it when we get done.

The complex integral

$$\int_\Gamma f \tag{5}$$

is a kind of line integral or contour integral, and you may have encountered real line integrals in Calculus III. I'm afraid, at the risk of being confusing, I need to mention also a different kind of integral we're going to use, namely, the integral of a complex function of one real variable. Of course, that's the same kind of function we used above to generate the curve Γ on which we want to integrate, but in this context, we think of such a function a bit differently, so I'll give it a different name. Fortunately, these kinds of integrals are really simple (to understand); they're essentially just like real integrals...sort of. If we have $g : (a, b) \to \mathbb{C}$, then remember $g = g_1 + ig_2$ has real and imaginary parts. In this case, we **define**

$$\int_a^b g(t)\,dt = \int_a^b g_1(t)\,dt + i \int_a^b g_2(t)\,dt.$$

This will be a complex number, but the real and imaginary parts of this complex number are just regular 1-D calculus integrals because $g_1 : (a, b) \to \mathbb{R}$ and $g_2 : (a, b) \to \mathbb{R}$. So these integrals are very simple, and there is not much to say about them. We might call such an integral a **complex valued integral**, but it's not really a **complex integral** as in (5). We'll define the new complex integral from (5) in terms of this kind of integral. The trick is that we need to have Γ parameterized by **arclength**.

Exercise 18 *Let $\gamma(t) = \cos t + i \sin t$ for $0 \le t \le \pi/4$. Draw the image curve and calculate the length of the curve between $\gamma(0)$ and $\gamma(t)$.*

Exercise 19 *Let $\alpha(t) = 3\cos t + 3i \sin t$ for $0 \le t \le \pi/4$. Find a parameterization γ of the image curve $\Gamma = \{\alpha(t) : 0 \le t \le \pi/4\}$ which is a parameterization of Γ by arclength. Hint: The domain of γ should be the interval $[0, 3\pi/4]$.*

The original parameterization we get may not be by arclength, but for a moment, let's assume it is. That is, let's assume $\gamma : (0, \ell) \to \mathcal{U}$ with

$$|\gamma'(t)| = 1 \qquad \text{for every } t. \tag{6}$$

The condition in (6) ensures that as t changes the point $\gamma(t)$ moves at **unit speed** in the complex plane. Consequently, the arclength

$$s = \int_0^t |\gamma'(t)|\,dt$$

is just the parameter t, and ℓ is the length of Γ. Perhaps I should point out here that the derivative $\gamma'(t)$ is **not** a complex derivative. It is the derivative analogue of our simple (complex valued) integral above:

$$\gamma'(t) = \frac{d}{dt}\operatorname{Re}[\gamma(t)] + i\frac{d}{dt}\operatorname{Im}[\gamma(t)].$$

If you understand all the players above, then we can simply write

$$\int_\Gamma f = \int_0^\ell f \circ \gamma(t)\, \gamma'(t)\, dt. \tag{7}$$

Notice the function $f \circ \gamma : (0, \ell) \to \mathbb{C}$ is a **composition** and has values given by $f \circ \gamma(t) = f(\gamma(t))$. Notice also that the value of the integral $\int_\Gamma f$ does not depend only on the curve Γ but also on the **direction** of the curve determined by the parameterization γ. Thus, a complex integral is always a **directed path integral**.

Exercise 20 *Explain the difference between the derivative of* $\gamma : \mathbb{R}^2 \to \mathbb{C}$ *by* $\gamma(x, y) = x + iy$ *and the derivative of* $f : \mathbb{C} \to \mathbb{C}$ *by* $f(z) = z$.

Exercise 21 *Given an arclength parameterization* $\gamma : (0, \ell) \to \Gamma \subset \mathbb{C}$ *of a curve* Γ *in the complex plane, find an arclength parameterization of* Γ *in the reverse direction, and explain what happens to the value of a complex integral* $\int_\Gamma f$ *if the direction of* Γ *is reversed.*

3.1 Interpretation: Real Line Integrals

Perhaps the easiest way to understand the definition (7) is, paradoxically, in terms of more general real valued integrals. You may have seen these in Calculus III, but they were probably not presented the way I am going to present them. A **real line integral**, like the complex integral above, is an integral of a real valued function $f : \Gamma \to \mathbb{R}$ defined on a curve Γ. Here is the definition:

$$\int_\Gamma f = \lim_{\|\mathcal{P}\| \to 0} \sum_j f(p_j^*)\ell_j. \tag{8}$$

This definition should remind you of the definition of a 1-D Riemann integral (from Calculus I) given as a limit of **Riemann sums**:

$$\int_a^b f(x)\, dx = \lim_{\|\mathcal{P}\| \to 0} \sum_{j=0}^{k-1} f(x_j^*)(x_{j+1} - x_j). \tag{9}$$

If you remember and understand (9), then (8) is an easy generalization. In both cases, one starts with a **partition** $\mathcal{P}$ of the domain. For (9) the partition is easy. It is just a partition of the interval $[a, b]$ which is a string of k distinct points ordered on the interval:

$$\mathcal{P} : \quad a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b. \tag{10}$$

The distance from one point to the next is, of course, $x_{j+1} - x_j$, which is a quantity you see in the Riemann sum and is called the **increment**. The **norm of the partition** is the largest of these lengths:

$$\|\mathcal{P}\| = \max_j (x_{j+1} - x_j).$$

If you need to review this kind of integral, it's now time for you to draw an illustration. The point x_j^* is called the **evaluation point** for the interval $[x_{j-1}, x_j]$, and it is located somewhere (anywhere) on this interval. Your picture should show that the **Riemann sum**

$$\sum_{j=0}^{k-1} f(x_j^*)(x_{j+1} - x_j)$$

is the sum of the areas of some rectangles with base lengths $x_{j+1} - x_j$ and "heights" $f(x_j^*)$ for each $j = 0, 1, \ldots, k-1$. I put "heights" in quotes because the value f(x_j^*) may be negative. This, of course, means that the "heights" of the rectangles are counted with a sign.

Perhaps one last comment on our "review" of the 1-D calculus Riemann integral: The limit is taken over all partitions and all possible choices of evaluation points. Technically, this works like this: The limit in (9) exists with the value $L \in \mathbb{R}$ if for any $\epsilon > 0$, there is some $\delta > 0$ such that $\|\mathcal{P}\| < \delta$ implies

$$\left| \sum_{j=0}^{k-1} f(x_j^*)(x_{j+1} - x_j) - L \right| < \epsilon$$

for any choice of the evaluation points $x_j \le x_j^* \le x_{j+1}$. If such a limit L exists, we call that number L the (definite) integral of f from a to b, and write

$$L = \int_a^b f(x)\, dx.$$

The real line integral (8) is a direct generalization of (9). Again, we begin with a partition. This time you either need to just imagine a partition of the curve Γ in your mind (or you can draw a picture), but if you want to write down an expression analogous to (10) you will need a parameterization, which the curve Γ may view as somewhat unneeded. Curves are a bit like nudists in this sense. In any case, if we are to dress our curve Γ with a parameterization, it might look like

$$\alpha : [a, b] \to \Gamma \subset \mathbb{R}^m.$$

Now we can write

$$\mathcal{P} : \quad \alpha(t_0), \alpha(t_1), \ldots, \alpha(t_k)$$

where

$$a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b \quad \text{is a partiation of } [a, b].$$

You should compare this to (10). The length ℓ_j associated with the piece of the curve between $\alpha(t_j)$ and $\alpha(t_{j+1})$ is even more difficult to express, but that length is easy to visualize. As long as the parameterization is differentiable, we can write a formula for ℓ_j:

$$\ell_j = \int_{t_j}^{t_{j+1}} |\alpha'(t)|\, dt.$$

You may remember that $\alpha'(t)$ is a tangent velocity vector to the curve giving the velocity with respect to the paraemeterization α, and the length $|\alpha'(t)|$ of this vector is the (instantaneous) speed of the parameterization. Consequently, we can write down the arclength more generally as

$$s = \int_0^t |\alpha'|\, dt,$$

and this almost always allows us to obtain an arclength paraemeterization $\gamma : [0, \ell] \to \Gamma \subset \mathbb{R}^m$ of the curve Γ with $\gamma = \gamma(s)$. Finally, the norm of such a partition is the largest of the numbers ℓ_j for $j = 0, 1, \ldots, k-1$, and the interpretation of such a line integral is pretty obvious. (Can you draw a picture, say for $\Gamma \subset \mathbb{R}^2$?)

3.1.1 Special Forms of the Integrand

We have just introduced the real line integral $\int_\Gamma f$. There are certain **forms** of the integrand $f : \gamma \to \mathbb{R}$ which are worth noting. Most of the forms require some additional information about Γ or that Γ have some special properties. First of all, it is usually required that Γ admits a **piecewise differentiable** parameterization $\alpha : (a, b) \to \Gamma$. This gives us, in particular, a well-defined tangent line to Γ at each point $\mathbf{p} \in \Gamma$ where $\gamma(t) = \mathbf{p}$ and $\gamma'(t) \neq 0$. Let us assume this kind of regularity condition, so that all but finitely many points on Γ determine a unique tangent line.

3.1.2 Real Flux Integrals

One piece of information we might have about Γ, or be given on Γ, in addition to the existence of tangent lines is a **unit normal field**. A unit normal field is a choice of a unit vector at each point of Γ orthogonal to the tangent line to Γ at that point. Let us denote a given unit normal field by $N : \Gamma \to \mathbb{R}^m$ by $N = N(\mathbf{p})$, assuming $\Gamma \subset \mathbb{R}^m$.

Say that, in addition to the unit normal field N, we have any vector field $\mathbf{v} : \Gamma \to \mathbb{R}^n$. Then we can consider the real valued function $f : \Gamma \to \mathbb{R}$ by $f(\mathbf{p}) = \mathbf{v}(\mathbf{p}) \cdot N(\mathbf{p})$ given by the Euclidean dot product of the vectors $\mathbf{v}$ and N at $\mathbf{p}$. Thus, we consider the integral

$$\int_\Gamma \mathbf{v} \cdot N \tag{11}$$

which is a line integral of a special form. If $\Gamma \subset \mathbb{R}^2$, then there are essentially two choices of continuous unit normal field N to Γ, and the special form integral (11) is called a **flux integral**. When $\mathbf{v}$ represents a **rate density** (of something flowing) across Γ, then the flux integral in (11) represents the rate of flow across Γ in the direction of N. What the something is that is flowing is a little vague here and depends strongly on the units of the field $\mathbf{v}$. You should think of several possibilities working out the units from the Riemann sum defining the integral. I'll give you a couple (common ones) to get started. In both these examples $\Gamma \subset \mathbb{R}^2$.

1. If $[\mathbf{v}] = M/T^3$ (Energy per time per length), then the flux integral $\int_\Gamma \mathbf{v} \cdot N$ gives the amount energy flowing across Γ in the direction of N per time, i.e., the energy flow rate.

2. If $[\mathbf{v}] = M/(LT)$ (Mass per time per length), then the the flux integral $\int_\Gamma \mathbf{v} \cdot N$ gives the mass flow rate across Γ.

The real flux integral appears in an important theorem from Calculus III which is usually called **Gauss' theorem for a plane curve**. It is also called the **divergence theorem in the plane**.

Theorem 6 *(Divergence Theorem in the Plane) If $\Gamma \subset \mathbb{R}^2$ is a simple closed plane curve and $\mathbf{v} = (v_1, v_2)$ is a vector field defined on an open set containing Γ and the subset $\mathcal{U}$ of the plane bounded by Γ, then*

$$\int_\Gamma \mathbf{v} \cdot N = \int_{\mathcal{U}} \operatorname{div} \mathbf{v} \tag{12}$$

where N is an **outward** *unit normal field on Γ and*

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}. \tag{13}$$

The integral on the right in (12) is yet another kind of integral. It is an **area integral** over the domain $\mathcal{U} \subset \mathbb{R}^2$, and the operator in (13) is called the **divergence** in rectangular coordinates. We will use this theorem about real integrals in a relatively simple way. You probably saw this theorem in Calculus III, but you might not remember it. Notice, first of all, that it is a kind of generalization of the **fundamental theorem of calculus** from Calculus I which says

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

That is, the integral of the derivative of a function over an interval is the value of the function evaluated on the boundary. The divergence theorem replaces the integral over the interval (a, b) with and integral over the domain $\mathcal{U}$ and replaces the derivative f' with the divergence of the vector field $\mathbf{v}$—a kind of derivative. It says that the integral of $\operatorname{div}\mathbf{v}$ over the domain $\mathcal{U}$ is the value of $\mathbf{v}$ "evaluated" on the boundary of $\mathcal{U}$, namely the curve Γ, in the sense of a flux integral over Γ.

In the special case $\operatorname{div}\mathbf{v} = 0$, the divergence therem simply says that the **net flow rate** out of Γ is zero, that is, when you have a divergence free field, you have to have the same rate of flow out as the rate of flow in, or conservation of the amount of that something (whatever it is) inside $\mathcal{U}$. This is the case we will use.

Exercise 22 *There is a version of the divergence theorem for a region $\mathcal{U} \subset \mathbb{R}^3$ (and even one for $\mathcal{U} \subset \mathbb{R}^n$). State the version for $\mathcal{U} \subset \mathbb{R}^3$. Hint: Increase the dimension on everything in the theorem by one.*

3.1.3 Real Circulation Integrals

Another special form arises when the curve Γ is a **directed curve**. In this case, instead of a normal field on Γ one has a **tangent field** which means that in addition to a **tangent line** at each point,[3] one has a **tangent vector** $T : \Gamma \to \mathbb{R}^m$. In this case, given a vector field $\mathbf{w} : \Gamma \to \mathbb{R}^m$ we define the **circulation integral** of $\mathbf{w}$ along Γ by

$$\int_\Gamma \mathbf{w} \cdot T.$$

This is really rather similar to a flux integral. In fact, $\int_\Gamma \mathbf{w} \cdot T$ **is** a flux integral for a different field. To see see this, note that it is possible to choose a **continuous plane field** along Γ consisting of two-planes containing the vectors $\mathbf{w}$ and T. Denote a rotation by 90 degrees in this plane field, i.e., in each plane of the plane field with the rotations continuously connected along the curve Γ, by a superscript "$\perp$." Then $N = T^\perp$ is a normal field on Γ and $\mathbf{v} = \mathbf{w}^\perp$ is a vector field along Γ for which the Euclidean products

$$\mathbf{w} \cdot T \qquad \text{and} \mathbf{v} \cdot N \qquad \text{are equal.}$$

[3]More precisely, I should say "except at finitely many points."

This means

$$\int_\Gamma \mathbf{w} \cdot T = \int_\Gamma \mathbf{w}^\perp \cdot T^\perp = \int_\Gamma \mathbf{v} \cdot N.$$

In the special case when $\Gamma \subset \mathbb{R}^2$, the plane field is simply the ambient $\mathbb{R}^2$ containing Γ and everything I have just described should be relatively easy to visualize. If we specialize even further to a simple closed curve Γ that is the boundary of a domain $\mathcal{U} \subset \mathbb{R}^2$, then we can get a version of the divergence theorem involving a circulation integral. It is called **Green's theorem**.

Theorem 7 *(Green's Theorem) If* $\Gamma \subset \mathbb{R}^2$ *is a simple closed plane curve and* $\mathbf{w} = (w_1, w_2)$ *is a vector field defined on an open set containing* Γ *and the subset* $\mathcal{U}$ *of the plane bounded by* Γ*, then*

$$\int_\Gamma \mathbf{w} \cdot T = \int_\mathcal{U} \left(\frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y} \right) \tag{14}$$

where T *is the* **counterclockwise** *unit tangent field on* Γ.

Proof (using the divergence theorem): Let N be the outward normal (field). Then the counterclockwise rotation of N by 90 degrees is the counterclockwise tangent field on Γ. That is, $N^\perp = T$. We can also define a field $\mathbf{v}$ by the condition $\mathbf{v}^\perp(-v_2, v_1) = \mathbf{w}$ so that

$$v_1 = w_2 \qquad \text{and} \qquad v_2 = -w_1.$$

This is because counterclockwise rotation of a vector $\mathbf{v} = (v_1, v_2)$ is obtained by switching the components and negating the (new) first one. (Try it with the standard unit vectors, and then note that this rotation/operation is a linear transformation, so it applies to all vectors the same.) Applying the divergence theorem to $\mathbf{v}$ gives

$$\int_\Gamma \mathbf{w} \cdot T = \int_\Gamma \mathbf{v} \cdot N = \int_\mathcal{U} \operatorname{div} \mathbf{v} = \int_\mathcal{U} \operatorname{div}(w_2, -w_1)$$

which is (14). $\square$

Arclength Parameterization

Very often one is given an entire parameterization $\alpha : (a, b) \to \Gamma \subset \mathbb{R}^n$. In this case the tangent vector giving a direction on Γ is $\alpha'(t)$. The parameterization can be exchanged, as usual, for another $\gamma : (0, \ell) \to \Gamma$ which is a parameterization by arclength.

Exercise 23 *If* $\alpha : (a,b) \to \Gamma$ *is a parameterization of a curve* $\Gamma \subset \mathbb{R}^m$ *and* $\gamma : (0,\ell) \to \Gamma$ *is a reparameterization of the same curve by arclength (in the same direction) wit* $\gamma(s)$ *corresponding to* $\alpha(t)$*, then what is the unit tangent vector* $\gamma'(s)$ *in terms of* $\alpha'(t)$*? You should go ahead and write down the relation between the arclength* s *and the parameter* t *and derive your expression for* $\gamma'(s)$ *using the chain rule.*

Exercise 24 *Given an arclength parameterization* $\gamma : (0,\ell) \to \Gamma$ *of a curve* $\Gamma \subset \mathbb{R}^2$*, show the following flux integral identity*

$$\int_\Gamma \mathbf{v} \cdot N = \pm \int_0^\ell \mathbf{v} \circ \gamma(t) \cdot \gamma'(t)^\perp \, dt.$$

What determines the sign?

Exercise 25 *Given an arclength parameterization* $\gamma : (0,\ell) \to \Gamma$ *of a curve, show the following circulation integral identity*

$$\int_\Gamma \mathbf{w} \cdot T = \int_0^\ell \mathbf{w} \circ \gamma(t) \cdot \gamma'(t) \, dt.$$

Exercise 26 *Given an* **any** *parameterization* $\alpha : (a,b) \to \Gamma$ *of a curve, show the integral identity for circulation*

$$\int_\Gamma \mathbf{v} \cdot T = \int_a^b \mathbf{v} \circ \alpha(t) \cdot \alpha'(t) \, dt$$

still holds.

3.2 Complex Integration

With this background of real integration along curves, we can see that the complex integral

$$\int_\Gamma f = \int_0^\ell f \circ \gamma(t) \, \gamma'(t) \, dt.$$

given in (7) may be viewed as an integral with a "special form" integrand. The integrand is a product involving the value of a complex valued function (somewhat similar to a field) and (something like a) tangent vector to the curve. In this case, the product is not a (real) Euclidean dot product, but rather the product of two complex numbers and, therefore, has both real and imaginary parts and is more, shall we say, complex.

Exercise 27 *Let $\Gamma \subset \mathbb{C}$ be a curve in the complex plane and $f : \Gamma \to \mathbb{C}$ a continuous functioned defined on Γ. Given any parameterization $\alpha : (a, b) \to \Gamma$ of Γ, what can you say about the values of*

$$\int_\Gamma f \qquad \text{and} \qquad \int_\Gamma f = \int_a^b f \circ \alpha(t)\, \alpha'(t)\, dt?$$

3.3 Theorems on Complex Integration

The first "big theorem" of complex integration is called **the Cauchy integral theorem**.

Theorem 8 *(Cauchy Integral Theorem) Let $f : \mathcal{U} \to \mathbb{C}$ is a complex differentiable (analytic) function in an open set $\mathcal{U} \subset \mathbb{C}$. If*

1. *Γ is a simple closed curve in $\mathcal{U}$ so that Γ is the boundary of another open set $\Omega \subset \mathbb{C}$, and*
2. *$\Omega \subset \mathcal{U}$,*

then

$$\int_\Gamma f = 0. \tag{15}$$

Proof (using the divergence theorem in the plane/Green's theorem): Taking a counterclockwise parameterization $\alpha = (x + iy) : (a, b) \to \mathbb{C}$ of Γ and writing $f = u + iv$ as usual, we have

$$\begin{aligned}\int_\Gamma f &= \int_a^b f \circ \alpha(t)\, \alpha'(t)\, dt \\ &= \int_a^b [ux' - vy' + i(uy' + vx')]\, dt \\ &= \int_a^b (ux' - vy')\, dt + i \int_a^b (uy' + vx')\, dt.\end{aligned}$$

We're going to use real integration theorems here, so let's set up a correspondence between $\mathbb{C}$ and $\mathbb{R}^2$ and introduce some new names. (We could use the same names, but let's try to make everything totally clear. You should draw pictures.) In particular, let's consider $\beta : (a, b) \to \mathbb{R}^2$ by

$$\beta(t) = (x(t), y(t))$$

and set

$$W = \{(x, y) \in \mathbb{R}^2 : x + iy \in \Omega\}.$$

Consider the real part of $\int_\Gamma f$, and note that the outward normal to the domain W is the counterclockwise rotation (by 90 degrees) of the real tangent vector $\beta' = (x', y')$, that is $N = (y', -x')/|\beta'|$. Thus, the integrand $ux' - vy'$ can be written using a real dot product as $\mathbf{v} \cdot N|\beta'|$ where $\mathbf{v} = (-v, -u)$. This means we can use the divergence therem and

$$\begin{aligned}\int_a^b (ux' - vy')\, dt &= \int_{\partial W} \mathbf{v} \cdot N \\ &= \int_W \operatorname{div} \mathbf{v} \\ &= -\int_W (v_x + u_y).\end{aligned}$$

According to the Cauchy-Riemann equations $u_y = -v_x$, so the real part of $\int_\Gamma f = 0$.

Let's apply Green's theorem to the imaginary part of $\int_\Gamma f$. We start with

$$\int_a^b (vx' + uy')\, dt = \int_a^b \mathbf{w} \cdot T\, |\beta'|\, dt = \int_{\partial W} \mathbf{w} \cdot T$$

where $\mathbf{w} = (v, u)$ and $T = \beta/|\beta'|$. Applying Green's theorem,

$$\int_{\partial W} \mathbf{w} \cdot T = \int_W \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right).$$

Again, according to the Cauchy-Riemann equations $u_x = v_y$ so this expression vanishes as well. $\square$

Exercise 28 *Prove the Cauchy integral theorem using Green's theorem for the real part and the divergence theorem for the imaginary part.*

The next theorem is call the **Cauchy integral formula**.

Theorem 9 *(Cauchy integral formula) Let $f : \mathcal{U} \to \mathbb{C}$ is a complex differentiable (analytic) function in an open set $\mathcal{U} \subset \mathbb{C}$. If*

1. *Γ is a simple closed curve in $\mathcal{U}$ so that Γ is the boundary of another open set $\Omega \subset \mathbb{C}$, and*

2. *$z_0 \in \Omega \subset \mathcal{U}$,*

then

$$f(z_0) = \frac{1}{2\pi i} \int_{z\in\Gamma} \frac{f(z)}{z - z_0} \tag{16}$$

where we take Γ *in the* **counterclockwise** *direction.*

We have used a slightly different notation in the integral appearing in (16), but it is also a standard one. The subscript "$z \in \Gamma$" on the integral sign shows the variable of integration. An alternative would be

$$\int_\Gamma \frac{f(z)}{z - z_0}\, dz$$

where the suffix "dz" indicated integration with respect to the variable z. This latter notation is used by Boas. Note that the counterclockwise direction of Γ must be understood or stated explicitly. There are notations to indicate counterclockwise orientation for a line integral around a simple closed curve:

$$\oint_{z\in\Gamma} \frac{f(z)}{z - z_0} \qquad \text{and} \qquad \ointctrclockwise_{z\in\Gamma} \frac{f(z)}{z - z_0}.$$

To emphasize the appropriate dependence, I'm now going to change names in (16) and write

$$f(z) = \int_{\zeta\in\Gamma} \frac{f(\zeta)}{\zeta - z}.$$

Writing it this way, you can see (more or less) clearly how this formula is used to get some of the striking results on differentiation. In particular, one can move the value of z without touching Γ or the variable of integration ζ. This means an expression like the complex difference quotent can be written as follows

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{\zeta\in\Gamma} f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0} \right).$$

In fact, it follows that one can **differentiate under the integral** to obtain

$$f'(z) = \int_{\zeta\in\Gamma} \frac{f(\zeta)}{(\zeta - z)^2}. \tag{17}$$

You should be picturing here the point ζ moving in the curve Γ and always staying away from z (and maybe z_0) which stays fixed inside Ω. As a result none of these integrands have any singularity.

One thing to note about this observation and the formula (17) is that you can now prove complex regularity. The dependence of f' on z does not really depend on the values of f' but only on the z dependence in the integrand.

Exercise 29 *Differentiate the formula (17) repeatedly differentiating under the integral on the right to obtain a formula for the n-the derivative of f. (You should observe that it is surprising to have obtained a formula for high order derivatives of f depending only on the values of f and no lower order derivatives.)*

This observations essentially shows that once f has a single derivative, then f has derivatives of all orders. This does not hold for real valued functions because there is no Cauchy integral formula for real valued functions.

Furthermore, once you have values for all the derivatives at a point $z \in \mathcal{U}$, you can write down the power series for f and try to prove analyticity.

Exercise 30 *Try to expand the integrand in the Cauchy integral formula as a power series in z centered at a point $z_0 \in \mathcal{U}$ and then use termwise integration to write down a power series for f. Hint: Write the denominator as $\zeta - z = (\zeta - z_0) - (z - z_0)$.*

The final result we are going to consider is the **residue theorem**. This requires a little background. First, the theorem applies to complex differentiable functions with **isolated singularities**. An isolated singularity of the complex differentiable function $f : \mathcal{U} \to \mathbb{C}$ is a point $z_0 \in \mathbb{C}$ in the complex plane such that for some $r > 0$, the **punctured disk**

$$B_r(z_0)\backslash\{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$$

is a subset of $\mathcal{U}$, the set of differentiability for f. Furthermore, we are interested in functions $f : \mathcal{U} \to \mathbb{C}$ with isolated singularities which are **poles**. The function f has a pole at the isolated singularity z_0 if f admits an expansion of the form

$$f(z) = \frac{a_{-k}}{(z - z_0)^k} + \frac{a_{-k+1}}{(z - z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z - z_0} + \sum_{j=0}^{\infty} a_j (z - z_0)^j \tag{18}$$

for some k. The power series on the right is required to converge to a complex differentiable function f_0 on some neighborhood

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

so that

$$f_0(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j \qquad \text{on} \qquad B_r(z_0).$$

In this case, assuming $a_{-k} \neq 0$, we say f has a **pole of order** k at z_0. If the order of a pole is $k = 1$, then f is said to have a **simple pole**. If z_0 is an isolated singularity of f but is not a pole, i.e., does not admit an expansion of the form (18), then z_0 is called an **essential singularity**. The function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

Exercise 31 *Write down the* **Laurent series** *for* $f(z) = e^{1/z}$. *Hint: Plug in* $1/z$ *into the power series for the exponential function. Notice that this function with an essential singularity still has a residue, i.e., coefficient* a_{-1}. *Can you calculate*

$$\int_{\partial B_r(0)} f?$$

(Of course, we mean to take a counterclockwise direction on the countour.)

Exercise 32 *Show that all the coefficients*

$$a_{-k}, a_{-k+1}, \ldots, a_{-1}, a_0, a_1, a_2, \ldots$$

in an expansion of the form (18) are unique. (This one is not so easy.)

We're almost there. As one looks at the expression (18) one of the coefficients is special. It is a_{-1}. This coefficient is called the **residue** of f at z_0. Here is the theorem:

Theorem 10 *(Residue Theorem) Let* $f : \mathcal{U} \to \mathbb{C}$ *be a complex differentiable function with isolated singularties (poles) at points* $p_1, p_2, \ldots, p_\ell \in \mathbb{C}$. *If*

1. Γ *is a simple closed curve in* $\mathbb{C}$ *so that* Γ *is the boundary of another open set* $\Omega \subset \mathbb{C}$,
2. $\partial\Omega \subset \mathcal{U}$,
3. $p_1, p_2, \ldots, p_\ell \in \Omega \subset \overline{\mathcal{U}}$, *and*
4. $\Omega \cap \overline{\mathcal{U}} = \{p_1, p_2, \ldots, p_\ell\}$,

then

$$\frac{1}{2\pi i} \int_{z\in\Gamma} f(z) = \sum_{j=1}^{\ell} \operatorname{res}(f, p_j) \tag{19}$$

where $\operatorname{res}(f, p_j)$ *is the residue of* f *at* p_j.

The statement of this theorem involves a couple **topological** concepts which you may not know yet. They are relatively easy. The word **topological** means "having to do with open sets," and you know what it means for a set to be open. (If you don't, go back and remind yourself.) The *definition* of a set being **closed** is that its complement is open. So now you know what a closed set is. Every set has a **closure** which is the smallest closed set containing that set. That is, if A is a set, the intersection of all closed sets containing A, written

$$\overline{A} = \bigcap_{\substack{C \supset A \\ C \text{ closed}}} C$$

is a closed set, and it is the smallest closed set containing A.

Exercise 33 *Show that the closure of a set is closed.*

We've written ∂A before to denote the **boundary** of a set. Now we can define the boundary in general:

$$\partial A = \overline{A} \cap \overline{A^c}$$

where $A^c = \mathbb{C} \backslash A$ is the **complement** of A.

Here is an exercise which briefly goes over the topological requirements in the residue theorem, what they mean, and why they are there.

Exercise 34

1. *The objective of the condition $p_1, p_2, \ldots, p_\ell \in \Omega \subset \overline{\mathcal{U}}$ may be viewed as intending to make each of the isolated singularities $p_1, p_2, \ldots, p_\ell$ an isolated singularity for f with respect to Ω. This doesn't quite work because f is not defined on all of Ω. For example, f is not defined at $p_1, p_2, \ldots, p_\ell$. So, to be more precise, we want $p_1, p_2, \ldots, p_\ell$ to be isolated singularities of f when restricted to $\Omega \backslash \{p_1, p_2, \ldots, p_\ell\}$.*

   (a) *Try to show that f restricted to $\Omega \backslash \{p_1, p_2, \ldots, p_\ell\}$ is a well-defined complex differentiable function. (Hint: You will not be able to do this.)*

   (b) *Show that if the restriction of f to $\Omega \backslash \{p_1, p_2, \ldots, p_\ell\}$ were a well-defined complex differentiable function, then $p_1, p_2, \ldots, p_\ell$ would be isolated singularities for this function.*

2. *The condition $\Omega \cap \overline{\mathcal{U}} = \{p_1, p_2, \ldots, p_\ell\}$ rules out the possibility of extra pieces of $\partial \mathcal{U}$ from "appearing" in Ω. Go back and try to do part (a) again using this assumption.*

3.4 An Example

I'm now going to give an example. Actually, it's sort of a series of examples, but it starts as an example of Cauchy's theorem. The function $f : \mathbb{C} \backslash \{0\} \to \mathbb{C}$ given by

$$f(z) = \frac{e^{iz}}{z}$$

is analytic in the punctured plane with a simple pole at $z = 0$. The contour Γ consisting of a small semicircle $C_r = \{re^{it} : 0 \le t \le \pi\}$ in the upper half plane, a large semicircle $C_R = \{Re^{it} : 0 \le t \le \pi\}$ (also in the upper half plane) and the two real segments

$$\Sigma_\pm = \{t : r \le \pm t \le R\}$$

bounds a simply connected domain Ω on which the restriction of f is nonsingular, i.e., complex differentiable. Cauchy's theorem says then that the integral of f over Γ should vanish. Let's consider each part of the integral separately. First, the small semicircle, taken counterclockwise with respect to the contour Γ, should be taken clockwise as part of a circle around the origin $z = 0$, so we get

$$\begin{aligned}\int_{C_r} f &= \int_\pi^0 \frac{e^{ire^{it}}}{re^{it}}\, rie^{it}\, dt \\ &= i\int_\pi^0 e^{ire^{it}}\, dt.\end{aligned}$$

In principle, this integral might be (and probably is) difficult to compute. But I'm going to avoid computing it. I'm just going to compute the limit as r tends to zero. Notice that when r tends to zero, the complex number

$$re^{it}$$

also tends to zero. Since the complex function $g(z) = e^{iz}$ is regular at the origin, this means the value of the integrand in

$$\int_\pi^0 e^{ire^{it}}\, dt$$

is going to uniformly approach $e^0 = 1$ all around the circle, or perhaps more properly in this case, on the entire backwards interval $\pi \ge 0 \ge 0$. To express this a little more rigorously, given any $\epsilon > 0$, we know $|e^{ire^{it}} - 1| < \epsilon$ for $0 \le t \le \pi$ whenever r is small enough and so

$$\left|\int_\pi^0 e^{ire^{it}}\, dt - \int_\pi^0 1\, dt\right| \le \int_0^\pi |e^{ire^{it}} - 1|\, dt < \epsilon\pi.$$

This shows,

$$\lim_{r\to 0}\int_\pi^0 e^{ire^{it}}\, dt = \int_\pi^0 1\, dt = -\pi,$$

and

$$\lim_{r\searrow 0}\int_{C_r} f = -\pi i. \tag{20}$$

The segment Σ_+ contributes

$$\int_{\Sigma_+} f = \int_r^R \frac{e^{it}}{t}\,dt = \int_r^R \frac{\cos t}{t}\,dt + i\int_r^R \frac{\sin t}{t}\,dt.$$

While we are at it, let's consider the piece coming from Σ_-. Here we can use the parameterization $\gamma(t) = -R + t$ for $0 \le t \le R - r$:

$$\int_{\Sigma_-} f = \int_0^{R-r} \frac{e^{i(t-R)}}{t-R}\,dt = \int_{-R}^{-r} \frac{e^{iu}}{u}\,du = \int_{-R}^{-r} \frac{\cos t}{t}\,dt + i\int_{-R}^{-r} \frac{\sin t}{t}\,dt.$$

(We used the substitution $u = t - R$.)

Finally, we consider the large circle C_R. When we considered the small circle, the image of the small circle under the exponential was a small curve near $z = 1$. (You could draw this curve with mathematical software.) Notice that the large (semi) circle C_R will pass through many fundamental domains of the complex exponential when R is large, so the corresponding image for this integral will be much more complicated. Again, we will just take a limit.

$$\begin{aligned}\int_{C_R} f &= i\int_0^\pi e^{iRe^{it}}\,dt \\ &= i\int_0^\pi e^{iR\cos t - R\sin t}\,dt \\ &= i\int_0^\pi e^{-R\sin t}\,e^{iR\cos t}\,dt.\end{aligned}$$

Now we will use a fundamental inequality which applies to all kinds of integrals including complex integrals, namely,

$$\left|\int g\right| \le \int |g|.$$

Noting that $e^{-R\sin t}$ should be a very small (positive) real number when $R\sin t$ is large and $e^{iR\cos t} = \cos(R\cos t) + i\sin(R\cos t)$ is a complex number of unit modulus, we have

$$\left|\int_{C_R} f\right| \le \int_0^\pi e^{-R\sin t}\,dt.$$

Consideration of the function $R\sin t$ for $0 \le t \le \pi$ tells us that on most of the interval $R\sin t$ is very large and positive (when R is large). The remainder of the interval should be very small, suggesting (at least the guess) that

$$\lim_{R\nearrow\infty} \int_{C_R} f = 0. \tag{21}$$

Of course, what is happening on the small part of the interval is somewhat complicated, so we should use some explicit estimates. First of all, we can certainly write

$$\int_0^\pi e^{-R\sin t}\,dt \le 2\int_0^{\pi/2} e^{-R\sin t}\,dt.$$

Since $\sin t$ is monotone on the interval $[0,\pi/2]$, we can change variables using $u = \sin t$ so that $du = \sqrt{1-u^2}\,dt$ and

$$\int_0^{\pi/2} e^{-R\sin t}\,dt = \int_0^1 \frac{e^{-Ru}}{\sqrt{1-u^2}}\,du.$$

An explicit estimate can then be obtained as follows

$$\begin{aligned}
\int_0^{\pi/2} e^{-R\sin t}\,dt &= \int_0^{1/\sqrt{R}} \frac{e^{-Ru}}{\sqrt{1-u^2}}\,du + \int_{1/\sqrt{R}}^1 \frac{e^{-Ru}}{\sqrt{1-u^2}}\,du\\
&\le \int_0^{1/\sqrt{R}} \frac{1}{\sqrt{1-(1/R)}}\,du + \int_{1/\sqrt{R}}^1 \frac{e^{-\sqrt{R}}}{\sqrt{1-u^2}}\,du\\
&= \frac{1}{\sqrt{R-1}} + e^{-\sqrt{R}}[\sin^{-1}(1) - \sin^{-1}(1/\sqrt{R})]\\
&\le \frac{1}{\sqrt{R-1}} + (\pi/2)e^{-\sqrt{R}}.
\end{aligned}$$

This establishes (21). Returning to the assertion of Cauchy's theorem:

$$\int_{C_r} \frac{e^{iz}}{z} + \int_{\Sigma_+} \frac{e^{iz}}{z} + \int_{C_R} \frac{e^{iz}}{z} + \int_{\Sigma_-} \frac{e^{iz}}{z} = 0 \tag{22}$$

Taking the limits as $r \searrow 0$ and $R \nearrow \infty$ we see the sum of the integrals over Σ_- and Σ_+ tends to the interesting complex valued intgral with respect to a real variable:

$$\int_{-\infty}^\infty \frac{e^{it}}{t}\,dt = \int_{-\infty}^\infty \frac{\cos t}{t}\,dt + i\int_{-\infty}^\infty \frac{\sin t}{t}\,dt.$$

Combining this with our other limiting values (22) becomes

$$-\pi i + \int_{-\infty}^\infty \frac{\cos t}{t}\,dt + i\int_{-\infty}^\infty \frac{\sin t}{t}\,dt = 0$$

or

$$\int_{-\infty}^\infty \frac{\cos t}{t}\,dt = 0 \qquad \text{and} \qquad \int_{-\infty}^\infty \frac{\sin t}{t}\,dt = \pi.$$

The first integral is obvious because $\cos t/t$ is odd, but the second real integral is very difficult to compute any other way. Boas gives a discussion[4] leading to the same result in her Example 4 on pages 690-692.

Let's (finally) briefly consider the entire circle of radius r about $z = 0$ and apply the residue theorem. Since

$$\frac{e^{iz}}{z} = \frac{1}{z}\sum_{j=0}^{\infty}\frac{(iz)^j}{j!}\frac{1}{z} + \sum_{j=1}^{\infty}\frac{(iz)^j}{j!},$$

we know $\mathrm{res}(f, 0) = 1$.

We showed above that the integral over the top half C_r of $\partial B_r(0)$ limits to $-\pi i$. This means

$$\lim_{r\searrow 0}\int_{-C_r}\frac{e^{iz}}{z} = \lim_{r\searrow 0}\int_0^{\pi} e^{-r\sin t}\left[i\cos[r\cos t] - \sin[r\cos t]\right]\,dt = i\pi.$$

In particular,

$$\lim_{r\searrow 0}\int_0^{\pi} e^{-r\sin t}\cos[r\cos t]\,dt = \pi \qquad \text{and} \qquad \lim_{r\searrow 0}\int_0^{\pi} e^{-r\sin t}\sin[r\cos t]\,dt = 0.$$

We also know from the residue theorem that

$$\int_{\partial B_r(0)}\frac{e^{iz}}{z} = 2\pi i.$$

This means, in particular, that if C_r^* denotes the bottom half of $\partial B_r(0)$, then

$$\lim_{r\searrow 0}\int_{-C_r}\frac{e^{iz}}{z} = i\pi.$$

Note further that the integral over the bottom half can be written as

$$i\int_{\pi}^{2\pi} e^{ir\cos t - r\sin t}\,dt$$

so that changing variables using $t = 2\pi - u$, we get

$$i\int_0^{\pi} e^{r\sin u}e^{ir\cos u}\,du = \int_0^{\pi} e^{r\sin t}\left[i\cos[r\cos t] - \sin[r\cos t]\right]\,dt.$$

[4]She also refers to the discussion of the previous example and does not give all the details for that. I have given all the details.

Of course, this is easilly seen to have the same limiting value πi. In particular,

$$\lim_{r\searrow 0}\int_0^{\pi} e^{r\sin t}\cos[r\cos t]\,dt = \pi \qquad \text{and} \qquad \lim_{r\searrow 0}\int_0^{\pi} e^{-r\sin t}\sin[r\cos t]\,dt = 0.$$

This suggests an interesting question:

Exercise 35 **(a)** *Go ahead and prove*

$$\lim_{r\searrow 0}\int_{C_r^*}\frac{e^{iz}}{z} = 1\pi$$

using estimates.

(b) *Are the values of*

$$\int_{C_r}\frac{e^{iz}}{z} \qquad \text{and} \qquad \int_{C_r^*}\frac{e^{iz}}{z}$$

the same or different from finite positive r?

If you want to become skilled at using the residue theorem to calculate difficult real integrals, the collection of problems Boas has at the end of section 7 of Chapter 14 is a good start.