Lecture 1: Complex Numbers and a complex function (or two)

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1 Preliminaries/Prerequisites

The most important foundational concept required in the material below is that of a **function**. You really need to know and be familiar with the notion of a function from one set to another. Presumably, we don't need any discussion of sets to get started, but you should note that whenever you want to talk about a function, you need to have two sets in mind; these sets are the **domain** of the function and the **co-domain** or **set of values** of the function. By **set of values** we can mean two different things (which are closely related). As a start, let us simply take the **co-domain** to be some set in which the values of the function are a subset. The precise set of values taken by the function is called the **range**. The **range** is also properly described as the **set of values** of the function.

While we do not need to talk about sets, some notation for sets might merit some review. You can find the "curly bracket notation" for sets, or more properly the **descriptive notation for sets**, described in any reasonable calculus textbook (or book on set theory). An example should be adequate. If X is the domain of a function f and Y is the codomain of the function, or the set in which f takes its values, then we can specify the range using descriptive notation by

$$R = \{y \in Y : y = f(x) \text{ for some } x \in X\}.$$

Briefly: The curly brackets are starting and ending symbols for the description of the set R. The specificiation to the left of the colon offers a **name** for the elements, i.e., a general element, in the set to be described. Properly, a **universe** for the set being described should be given. In this case, the universe is Y, but sometimes the specification of the universe is omitted:

$$R = \{y : y = f(x) \text{ for some } x \in X\}. \tag{1}$$

As an aside, while not explicitly specifying a universe (which is a set) is okay, not **having** a universe which is a set can cause logical trouble. That is to say, the set of all sets, or

$$\{S : S \text{ is a set}\}$$

is not a set. As I wrote above, hopefully, we don't need to get into a discussion of sets, but for those who might be interested, you can think about this:

Exercise 1 *Letting $A = \{S : S \text{ is a set}\}$, consider the descriptive specification*

$$B = \{S \in A : S \notin S\}.$$

Can you determine if B is an element of B, i.e., is the statement $B \in B$ true, or is the statement $B \notin B$ true?

Returning to our discussion of the specified set R, the specification to the right of the colon is, perhaps, the most important thing to understand. This should be some kind of sentence which allows you to determine whether any particular element (of any set) is an element of the particular set being described, in this case the set R. The whole thing written in (1) is read:

> *R is the set of elements y in Y* **such that** *$y = f(x)$ for some $x \in X$.*

An alternative to the descriptive notation is the **list notation** which also uses curly brackets:

$$C = \{0, 1, 2, 7\} \qquad \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

The first of these is read

> *C is the set consisting of the elements 0, 1, 2, and 7.*

In this case, the universe can be, for example the set of integers $\mathbb{Z}$, which is specified as the set $\mathbb{Z}$ containing 0, ± 1, ± 2, ± 3, **and so on**. These sets also need universes, which we have not been careful to specify.

Exercise 2 *In set theory, one starts with an axiom (the Axiom of Existence) asserting the existence of* **some** *set and then proves, using descriptive specification, the existence of a special set called the empty set ϕ containing no elements. Can you prove the existence of the empty set and build from that a suitable universe for the integers? (Caution: You will need the* **Axiom of Unions** *which states that if you have "two" sets A and B, then there exists a "third" set containing these "two" sets A and B as elements. The words "two" and "third" are in quotes here because at this stage in the construction of the integers using sets, one can not assume there is any such thing as 2 or 3. This exercise may well be beyond/below what you want to spend time on.)*

Let's say that's enough on set notation and move on to functions.

Say you have a domain X and co-domain Y. Informally, a **function** $f : X \to Y$, read f is a function from X to Y, is **a rule or correspondence which assigns to each** $x \in X$ **a unique** $y \in Y$. This is the important abstract concept you need to understand (fully). The reason it is informal, is because the words "rule" and "correspondence" have not been carefully defined. The formal definition is something like this:

> Start with the **Axiom of Products** which states that given two sets X and Y, there exists a third set $P = X \times Y$ consisting of all **ordered pairs** (x, y) such that $x \in X$ and $y \in Y$. That seems reasonable enough.
>
> A **relation** is any subset of the product $X \times Y$.
>
> A **function** F is a relation with the property that for each $x \in X$
>
> $$\{y \in Y : (x, y) \in F\}$$
>
> consists of exactly one element.

Exercise 3 *Writing $y = f(x)$ whenever $(x, y) \in F$ (and F is a function with domain X and range contained in Y), convince yourself that the informal definition of a function and the formal one are equivalent.*

Incidentally, the definition of a function I have just given you seems to have been first precisely understood by Leonhard Euler in the early eighteenth century. I mention this simply to emphasize the fact that it took a long time for any human to understand precisely the concept of a function, though probably many humans had understood it intuitively (albeit imprecisely) long before that. You have probably had an intuitive, though imprecise, understanding of the notion of a function for a long time, and now you need to understand that notion precisely.

As a last preliminary/prerequisite you should know about and be (somewhat) familiar with the set of **real numbers** $\mathbb{R}$. In particular, you should have in your mind the geometric interpretation of this set as a **line** with no gaps and be able to talk about, understand, and competently use **limits** of real numbers with relative ease.

2 Complex Numbers

A **complex number** is often denoted by the letter z, and we write $z = x + iy$ or $z = a + bi$ where a, b, x and y are **real numbers**. Say $x, y \in \mathbb{R}$ and $z = x + iy$. Then we can write

$$\begin{aligned} x &= \mathrm{Re}(z) \qquad \text{(the real part of } z) \\ &\text{and} \\ y &= \mathrm{Im}(z) \qquad \text{(the imaginary part of } z). \end{aligned}$$

Recall that the real line $\mathbb{R}$ (or the real numbers) are pictured as a line. Similarly, the set of complex numbers $\mathbb{C}$ is pictured as a plane with the complex number $z = x + iy$ having coordinates x and y, or more properly horizontal coordinate x in the real line (x-axis) as usual and vertical coordinate iy in the imaginary axis:

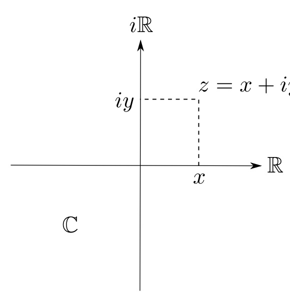

Figure 1: A complex number in the complex plane

Associated with each complex number $z = x + iy$ there is a **modulus** defined by

$$r = |z| = \sqrt{x^2 + y^2}. \qquad (2)$$

The modulus is also called the *polar radius* of z, and it should be obvious why that is the case.[1] If the complex number $z = x + iy$ is nonzero, then one can also assign to z an **argument**

$$\theta = \mathrm{Arg}(z) \tag{3}$$

which is an angle determined up to an additive integer multiple of 2π:

$$\theta = \mathrm{Arg}(z) + 2\pi k, \qquad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

It should be noted that the polar radius in (2) is given in terms of an explicit formula in terms of x and y while the argument in (3) is not. Here is an exercise that may be helpful as motivation for the material covered below.

Exercise 4 *Try to express the argument* $\mathrm{Arg}(z)$ *in terms of* $z = x + iy$.

3 The complex tangent function

A first attempt at giving a formula for $\mathrm{Arg}(z)$ might look like this:

$$\mathrm{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right). \tag{4}$$

And what one would probably mean by "$\tan^{-1}$" here is the function whose graph is illustrated in Figure 2. This function is sometimes called the **principal arctangent function** and is also denoted by "arctan" or "Tan^{-1}." It will be noted that there could be other choices for the inverse tangent as illustrated in Figure 3. These other choices are referred to as different **branches** of the inverse tangent function. The terminology derives from complex analysis which offers a unified framework in which these different real branches of the inverse tangent are tied together in a useful and beautiful manner. Probably most of you are not too familiar with this framework of complex function theory, so we are going to take some time to go through it carefully. That will be the main objective for the rest of this lecture, and the first step is to understand the extension of the real valued tangent function as a function of a complex variable.

You may also think of the following discussion being motivated by the following idea: The real arctangent function is useful in writing down a formula for the argument of a complex number sometimes, so perhaps the complex tangent (and the complex

[1] This might be a good time to do Problem 1.

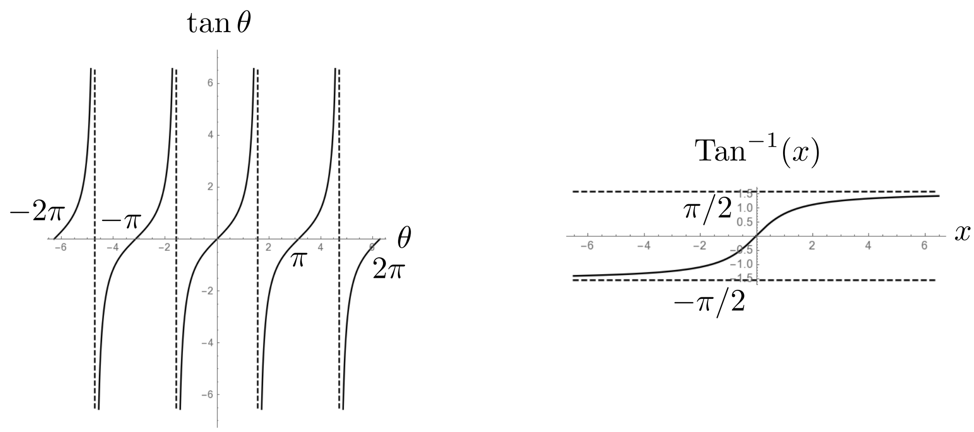

Figure 2: The real tangent function and the principal inverse tangent.

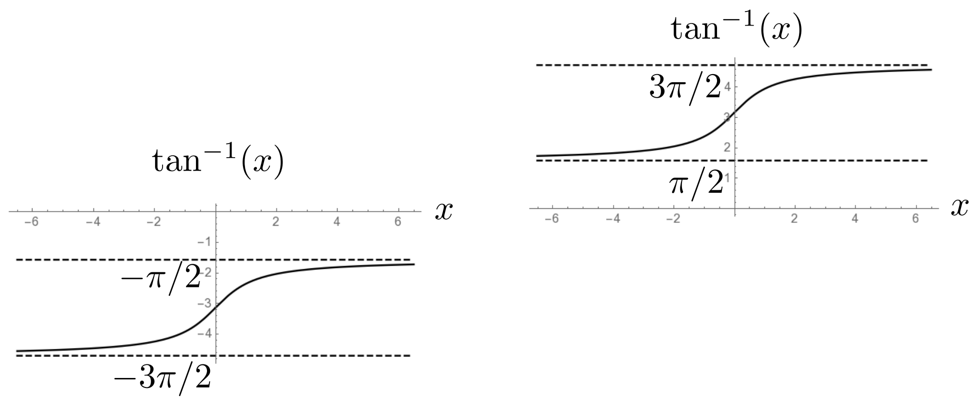

Figure 3: Other real inverse tangent functions.

inverse tangent) can be used to write down a formula for the argument of a complex number z in a more general way.

Before we discuss the value of $\tan z$ where $z = x + iy \in \mathbb{C}$, let's note the shortcoming of the formula given in (4) and see one solution of Exercise 4. In fact, if $\mathrm{Re}(z) = x > 0$, then our interpretation of $\tan^{-1}(y/x)$ as the principal inverse tangent gives the correct value of $\mathrm{Arg}(z)$. But the principal arctangent, as we know, can only give values strictly between $-\pi/2$ and $\pi/2$. Therefore, the formula in (4) gives an incorrect answer when $\mathrm{Re}(z) = x < 0$ and gives no answer at all when $\mathrm{Re}(z) = 0$. A correct answer to the exercise might be

$$\operatorname{Arg}(z) = \begin{cases} \arctan(y/x), & \operatorname{Re}(z) > 0 \\ \pm\pi/2, & \operatorname{Re}(z) = 0,\ \pm\operatorname{Im}(z) > 0 \\ \arctan(y/x) \pm \pi, & \operatorname{Re}(z) < 0 \\ \text{undefined}, & z = 0. \end{cases}$$

It is difficult to get away from the nominal unpleasantness of having to use cases to give this formula. On the other hand, the different branches of the inverse tangent can also be somewhat more systematically organized as follows: Note that the vertical asymptotes at $\theta = \pi/2 + k\pi$ for $k = 0, \pm1, \pm2, \pm3, \ldots$ divide the real line into intervals I_k as indicated in Figure 4. For each k, furthermore, we may set $\tan_k^{-1}\theta = \tan_0^{-1}\theta + k\pi$ where $\tan_0^{-1}$ denotes the principal arctangent. Then our formula may be written in

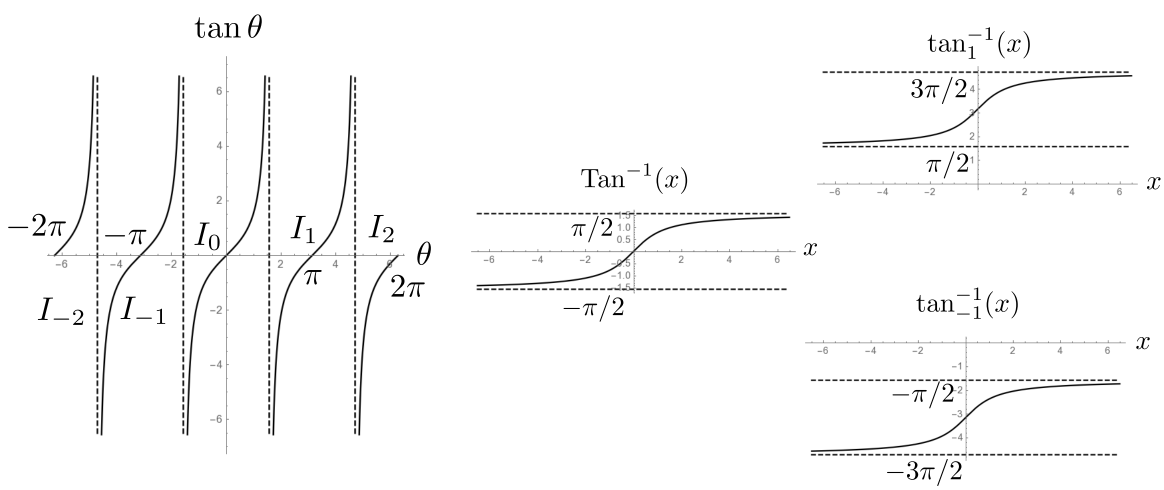

Figure 4: The real tangent function and the real branches of inverse tangent.

a somewhat more organized manner as

$$\operatorname{Arg}(z) = \begin{cases} \tan_0^{-1}(y/x), & \operatorname{Re}(z) > 0 \\ \pm\pi/2, & \operatorname{Re}(z) = 0,\ \pm\operatorname{Im}(z) > 0 \\ \tan_1^{-1}(y/x), & \operatorname{Re}(z) < 0 \\ \text{undefined}, & z = 0. \end{cases}$$

Finally, let us turn to the complex tangent, and see if we can connect these real branches of the inverse tangent. Not surprisingly, a natural approach to finding the value of $\tan z = \tan(x + iy)$ where x and y are real is to start with

$$\tan z = \frac{\sin z}{\cos z}$$

where

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{5}$$

If these formulas are not familiar to you, then note them for now (maybe even memorize them for now) and we will come back to them. These definitions are given in terms of **complex exponentials** which can be computed without too much trouble. For example,

$$e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}(\cos x + i \sin x). \tag{6}$$

Before we go on, let's think a little bit about what happened here. For any complex number $z = x + iy$, we can define the complex exponential as a series:

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}. \tag{7}$$

This should be familiar, at least in the case when $z = x \in \mathbb{R}$ is real. Also in the complex case, there is a relatively simple theory of convergence of such series, and because the factorials in the denominators grow so quickly with j, the series will always converge to a well-defined complex number for any $z \in \mathbb{C}$. For those who want to know precisely what this means, it means the following:

> *There exists a unique complex number w such that for any positive (real) number ϵ, there is some positive integer N for which*
>
> $$\left| w - \sum_{j=0}^{k} \frac{z^j}{j!} \right| < \epsilon \qquad \textit{whenever } k > N.$$

We call the number w the **value of the infinite series** and, in this case, we write $w = e^z$. In any case, we can use (7) as a starting point for consideration of the complex exponential e^z, and the rest follows from that. For example, if we take $z = ix$ to be purely imaginary, then

$$e^{ix} = \sum_{j=0}^{\infty} \frac{(ix)^j}{j!} = \sum_{\ell=0}^{\infty} \frac{(ix)^{2\ell}}{(2\ell)!} + \sum_{\ell=0}^{\infty} \frac{(ix)^{2\ell+1}}{(2\ell+1)!}.$$

Here we have just broken the sum up into the terms with even indices and the terms with odd indices. Noting that

$$(ix)^{2\ell} = (i^2)^\ell x^{2\ell} = (-1)^\ell x^{2\ell} \qquad \text{and} \qquad (ix)^{2\ell+1} = i(-1)^\ell x^{2\ell+1},$$

we get

$$\begin{aligned} e^{ix} &= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell x^{2\ell}}{(2\ell)!} + i \sum_{\ell=0}^{\infty} \frac{(-1)^\ell x^{2\ell+1}}{(2\ell+1)!} \\ &= \cos x + i \sin x. \end{aligned}$$

This result is worth noting; it is called Euler's formula:

$$e^{ix} = \cos x + i \sin x, \qquad x \in \mathbb{R}. \tag{8}$$

We used Euler's formula above[2] to see that

$$e^{iz} = e^{-y}(\cos x + i \sin x).$$

It follows similarly that

$$e^{-iz} = e^{y-ix} = e^y(\cos x - i \sin x).$$

Therefore,

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ &= \frac{1}{2i}[e^{-y}(\cos x + i \sin x) - e^y(\cos x - i \sin x)] \\ &= \frac{1}{2i}[(e^{-y} - e^y)\cos x + i(e^{-y} + e^y)\sin x \\ &= \sin x \cosh y + i \cos x \sinh y. \end{aligned}$$

In Problem 5 you are asked to make a similar calculation to obtain

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

If you are not familiar with the hyperbolic cosine and hyperbolic sine functions, Problem 6 should be of interest.

[2] This is a good time to do Problem 3 and Problem 4.

Using our expressions for $\cos z$ and $\sin z$, we have

$$\tan z = \frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y} = \frac{\cos x \sin x + i \cosh y \sinh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}.$$

I've used here the identities $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$.

Now, here is an important point you need to understand: The function tan, as we have expressed its values, is a function with domain $\mathbb{C}$ and co-domain $\mathbb{C}$. That means we have a **mapping from one plane to another**, and we want to understand how that mapping works. So, you need to start picturing in your mind two planes, one as the domain of the complex tangent function and the other where that function takes its values. See Figure 5.

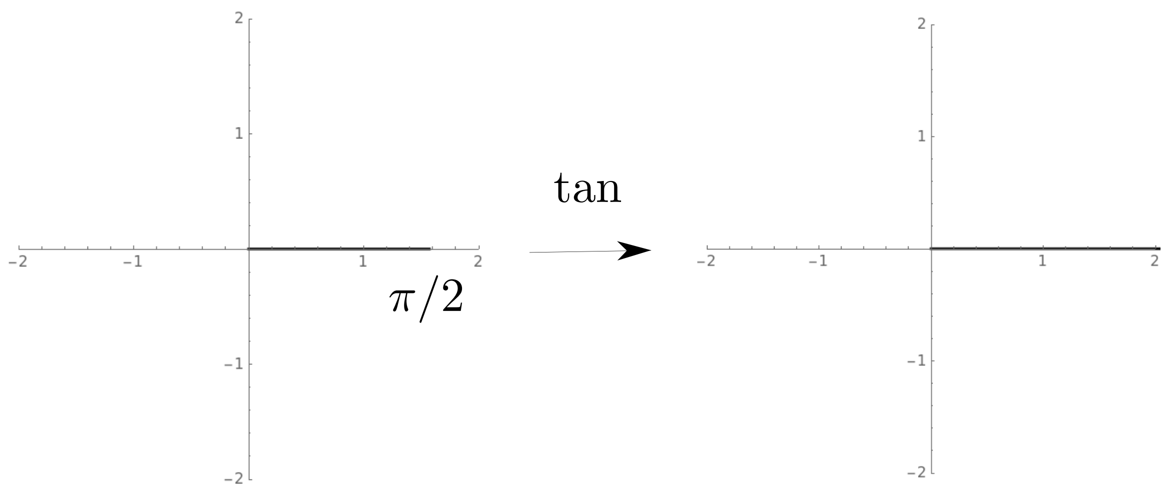

Figure 5: The domain and co-domain of the complex tangent function.

The way this mapping works is determined by the real and imaginary parts of the image. These are two (real valued) functions of two real variables. Writing $z = x + iy$ as above and $w = \tan z = \xi + i\eta$, we have

$$\xi(x, y) = \operatorname{Re} w = \frac{\cos x \sin x}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

and

$$\eta(x, y) = \operatorname{Im} w = \frac{\cosh y \sinh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}.$$

As a first step toward understanding the map, we can recover what we know about the real tangent function. Taking $y = 0$, so that $z = x + iy = x$ is real, we get

$$\xi(x,0) = \tan x \qquad \text{and} \qquad \eta(x,0) = 0.$$

That is the image point $\tan z = \xi(x,0) + i\eta(x,0) = \tan x \in \mathbb{R}$ corresponds to the point with coordinates $(\tan x, 0) \sim \tan x + 0i$ in the plane. You should see this in Figure 5 where we have shaded

$$\{x \in \mathbb{R} : 0 \le x < \pi/2\} \subset \mathbb{C}$$

in the domain on the left and (the visible part of)

$$\{\xi \in \mathbb{R} : \xi \ge 0\} = \text{ the positive real axis}$$

in the codomain. In fact, you should see clearly how the complex tangent mapping behaves when restricted to the real interval $-\pi/2 < x < \pi/2$. We know, furthermore, that this behavior will repeat on every interval of length π starting at any point $-\pi/2 + \pi k$ for k and integer. Let me suggest that you pause here for a moment and consider carefully the representation of the values of $\tan x$ illustrated by the (real) graph in Figure 2 as they appear in Figure 5. In particular, you should see an **expansion** in Figure 5 in which the interval $-\pi/2 < x < \pi/2$ is stretched and stretched so drastically that the points $x = \pm\pi/2$ correspond to ∞ in the complex plane. This expansion to infinity is represented by the vertical asymptotes in Figure 2, and I think it's important for you to understand what is happening here fully before you continue.

As a second step, I suggest considering the restriction of $\tan z$ to the imaginary axis. This means we'll take $z = x + iy = iy$ with $x = 0$. Computing the values of ξ and η, we find

$$\tan(yi) = i \tanh y. \tag{9}$$

The hyperbolic tangent is an interesting function. It is very different from the real tangent. You might use numerical software if you are not familiar with this function to see that it's graph resembles that of the principal arctangent illustrated in Figure 2. Instead of the asymptotic values $\pm\pi/2$ at $\pm\infty$, however, tanh tends to ± 1. The result is that along the imaginary axis the complex tangent mapping is a **compression** or **contraction**. The compression furthermore is so radical that the entire imaginary axis is compressed into the finite interval between $-i$ and $+i$ with $-\infty i$ mapping to $-i$ and $+\infty i$ mapping to i. What we have described is telling us something new and important about the tangent function which one does not know if one only knows the real tangent function. This compression for purely imaginary arguments (and values) is inherent to the complex tangent function.

If you see clearly the behavior we have described on the axes within the vertical strip

$$(-\pi/2, \pi/2) \times i\mathbb{R} = \{x + iy : -\pi/2 < x < \pi/2\}$$

then picture both together: A large stretching along the real axis taking $(-\pi/2, \pi/2)$ to the entire real line and a contraction of the entire imaginary axis to the interval between $-i$ and i. Now, you should ask yourself what happens in the rest of the the complex plane, i.e., what does tan do with the rest of $\mathbb{C}$? As a first step to understanding the answer to that question we can start with points in the first quadrant or at least the portion of the strip in the first quadrant. Maybe we could consider the line $\pi/2 + yi$ for $y > 0$. In view of the singularity at $z = \pi/2$ one might expect singular behavior. That, however does not turn out to be the case:

$$\tan(\pi/2 + iy) = i \coth y.$$

The hyperbolic cotangent given by $\coth y = \cosh y / \sinh y$ does indeed have a singularity at $y = 0$, but we are looking now at $y > 0$. Thus, something interesting happens. The half line $\{\pi/2 + iy : y > 0\}$ is mapped to the ray $\{i\eta : \eta > 1\}$ on the imaginary axis with a **reverse orientation**. That is, ∞ at the top end of the half line maps to i and the starting point $\pi/2$ on the real axis maps to ∞.

Now I'm going to suggest something which may not be obvious. I suggest we break the strip up into vertical and horizontal lines. As an example, let's take a horizontal line segment $\{x + iy : 0 \le x < \pi/2\}$ with $y > 0$ fixed. If we plot the image of this segment with mathematical software (like Mathematica) we see something like Figure 6.

In the picture it looks like the point $\xi + i\eta$ for $x + iy$ in the horizontal segment $\{x + iy : 0 \le x < \pi/2\}$ lies on a circle with center on the imaginary axis. In fact, we can check directly that

$$\tan(yi) = i \tanh y \qquad \text{and} \qquad \tan(\pi/2 + yi) = i \coth y.$$

Notice that these pure imaginary numbers will both have positive imaginary parts. This means that if our guess (based on the picture) is correct, then the center should

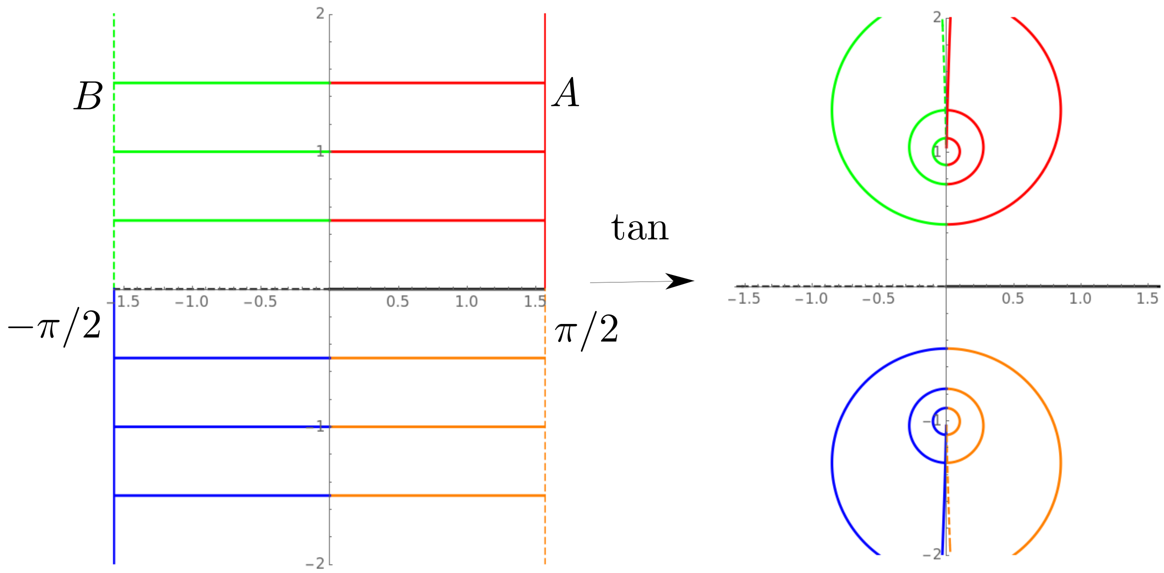

Figure 6: A carefully chosen subdomain of the complex tangent function and its image.

be at the point with imaginary part the average

$$\begin{aligned}\frac{\tanh y + \coth y}{2} &= \frac{1}{2}\left(\frac{\sinh y}{\cosh y} + \frac{\cosh y}{\sinh y}\right)\\ &= \frac{\cosh^2 y + \sinh^2 y}{2\cosh y \sinh y}.\end{aligned}$$

That is, the center should be

$$c = \frac{\cosh^2 y + \sinh^2 y}{2\cosh y \sinh y}\, i.$$

We can also calculate the guessed radius:

$$\begin{aligned}r &= \frac{1}{2}\left(\frac{\cosh y}{\sinh y} - \frac{\sinh y}{\cosh y}\right)\\ &= \frac{1}{2\cosh y \sinh y}\\ &= \operatorname{csch}(2y).\end{aligned}$$

In fact, this guess is correct. If you understand this assertion, you can get a pretty good idea of how the complex tangent function works. To strengthen and expand your understanding, Problem 8 below and contemplation of Figure 7 should be helpful.

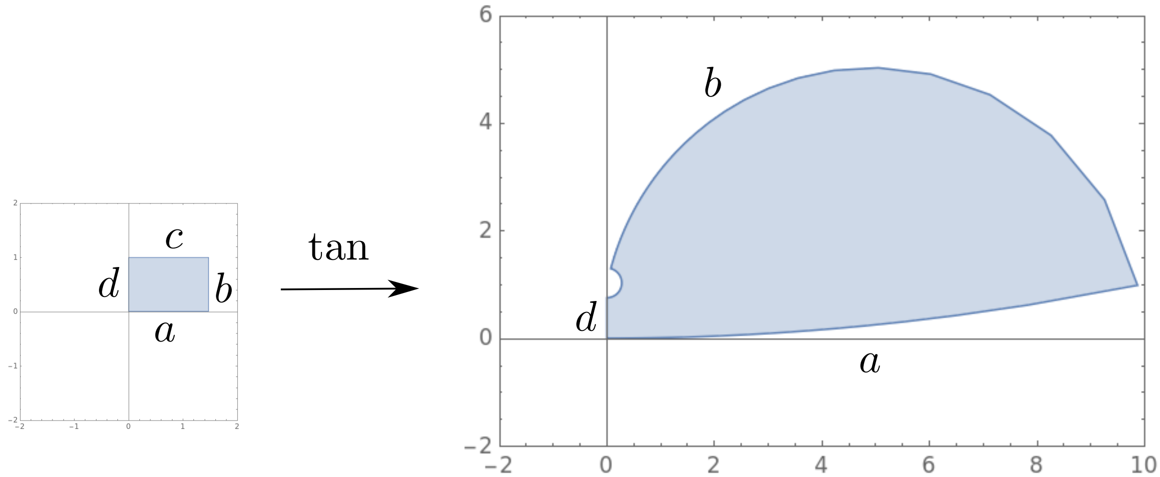

Figure 7: Another particular subdomain of the complex tangent function and its image. The domain here is the rectangle $[0, \pi/2 - 0.1] \times [0.01, 1]$. The image is called a **conformal rectangle**. The sides of the conformal image are not straight lines.

Exercise 5 *Make plots of the domain and image for various rectangles like that shown in Figure 7. In particular, note what happens when the interval for the imaginary part is* $[0, 1]$ *instead of* $[0.01, 1]$*. Why did I choose* 0.01 *as the lower limit?*

You should come to the conclusion that the strip

$$\Sigma = \{x + iy : -\pi/2 \le x \le \pi/2\} \setminus (\{-\pi/2 + yi : y \ge 0\} \cup \{\pi/2 + yi : y \le 0\}),$$

which we have modified by removing certain selected boundary portions, maps in a one-to-one manner exactly onto the entire complex plane with the exception of the points $\pm i$. Let that sink in. This means there is a well-defined inverse function

$$\tan^{-1} : \mathbb{C} \setminus \{\pm i\} \to \Sigma.$$

Notice how this function contains the real values of the real principal arctangent $\tan_0^{-1}$, but this function contains a tremendous amount of additional information. We can also, relatively easily see how this function extends to other branches, but to properly understand that, we need a new structure.

4 Riemann Surfaces

Were we to imagine a point moving in the domain of the complex **inverse** tangent function discussed above and passing from the first quadrant to the second across imaginary axis above the singular value i we know the image crosses the boundary of the strip Σ along the vertical segment

$$A = \{\pi/2 + yi : y > 0\}.$$

In contrast, if our point approaches (and crosses) the same segment from the opposite side in the second quadrant, then the image crosses the quite different vertical segment $B = \{-\pi/2 + yi : y > 0\}$. In both cases, we can see, more or less, how the extension of the inverse tangent function should work. To be very specific, if we continue moving on one of the circular arcs centered on the imaginary axis and pictured on the right in Figure 6, then when the point returns to the imaginary axis at a point $\eta_1 i$ lying between 0 and i, i.e., with positive imaginary part η_1. The image of the point $\eta_i i$ (under the inverse tangent) will lie at a point with positive imaginary part and real part π. To convince yourself this is the case, you might calculate $\tan(\pi + yi)$.

Admittedly, this should be a little disturbing to you. The reason this should be disturbing is because we have already discussed the behavior of the complex tangent function on a purely imaginary argument yi.

Exercise 6 *Given a purely imaginary value $\eta_1 i$ with $0 < \eta_1 < 1$, find a purely imaginary number yi such that $\tan(yi) = \eta_1 i$. Hint: Look at (9).*

These apparent contradictory observations should be made clear in just a moment. Let us, however, continue our journey constructing the inverse tangent (without the proper Riemann surface) a little further.

Let our same point (tied to its image under $\tan^{-1}$) move down the imaginary axis from $\eta_i i$ to the symmetric (conjugate) point $-\eta_1 i$. You can check directly that the image under $\tan^{-1}$ will move to a symmetric (conjugate) point on the line $\operatorname{Re} z = \pi$. Then moving on the symmetric circular arc in the third quadrant, the image under $\tan^{-1}$ will arrive back to the line $\operatorname{Re} z = \pi/2$ at a point with negative imaginary part. From there, as we cross the imaginary axis again, from the third to the fourth quadrant, and return to familiar territory with image in the strip Σ.

We have taken a curious journey, evidently **out** of the pictured domain of $\tan^{-1}$ on the right of Figure 6 and then back in again. But where did we go? The key is to construct the proper domain for the inverse tangent. Each strip, roughly determined by the condition $-\pi/2 + \pi k < \operatorname{Re} z < \pi/2 + \pi k$ for $k \in \mathbb{Z}$ maps under the tangent function to a copy of the complex plane $\mathbb{C}$. Thus, the image of the complex tangent is infinitely many copies of the complex plane with two very important qualifications/modifications.

First, each copy of the complex plane has two points omitted. These points are $\pm i$. And second, these copies are "sewn together" is a precise manner to produce something that looks rather like a surface, but not quite like a surface you are very used to seeing. The standard way to describe this sewing of the copies is to imagine a **branch cut** extending along the imaginary axis from i to $+\infty i$ and another from $-i$ to $-\infty i$. The points $\pm i$ are called **branch points**. Along each branch cut, we specify one edge to be "in" the copy, and the other to be "out" of the copy. If you look carefully at Figure 6 you should be able to see that I've indicated one edge of each branch cut as being "in" by drawing it with a solid line, and the opposite edge is designated as "out" because it is dashed.

A little more terminology and an identification associated with Figure 6 may be helpful. Each strip, for example the strip $\Sigma_0 = \Sigma$ shown in Figure 6 is called a **fundamental domain** for the tangent. The image of a fundamental domain, sometimes called a **fundamental region** on the Riemann surface is something like a full copy of the complex plane. More precisely, the fundamental region is a copy of the complex plane with some branch points and branch cuts. Let us denote the fundamental domain associated with the strip $-\pi/2 + \pi k \leq \operatorname{Re} z \leq \pi/2 + \pi k$ by the symbol Σ_k. These regions are shown on the left in Figure 6. Now, the image of Σ_k in the complex plane $\mathbb{C}$ is shown on the right in Figure 6. When we construct the Riemann surface, however, we take a different fundamental region for each strip Σ_k. Let us denote the fundamental region (in the Riemann surface) associated with Σ_k by $\mathcal{R}_k$. Then you should interpret Figure 6 as only showing the fundamental region $\mathcal{R}_0$ on the right. In Figure 8 we have two fundamental regions $\mathcal{R}_0$ and $\mathcal{R}_1$ shown.

Exercise 7 *The edges of the branch cuts in $\mathcal{R}_1$ with arrows on the right in Figure 8 could be labeled as "sewn to $\mathcal{R}_0$ (first quadrant)" (red) and "sewn to $\mathcal{R}_0$ (fourth quadrant)" (orange). Can you correctly label the two black edges of the branch cuts in $\mathcal{R}_1$?*

As a final effort to illustrate the Riemann surface, I'm going to bend and stretch the fundamental regions a little bit (and show how the fundamental regions are sewn together). You should be able to see in Figure 8 why Riemann surfaces are often said to have a "parking deck structure." One aspect which is rather distorted in this illustration (and rather difficult to illustrate accurately in general) is the fact that we now have two vertical lines over the branch points $\pm i$. In the Riemann surface, these branch points constitute only two, and exactly two, points.

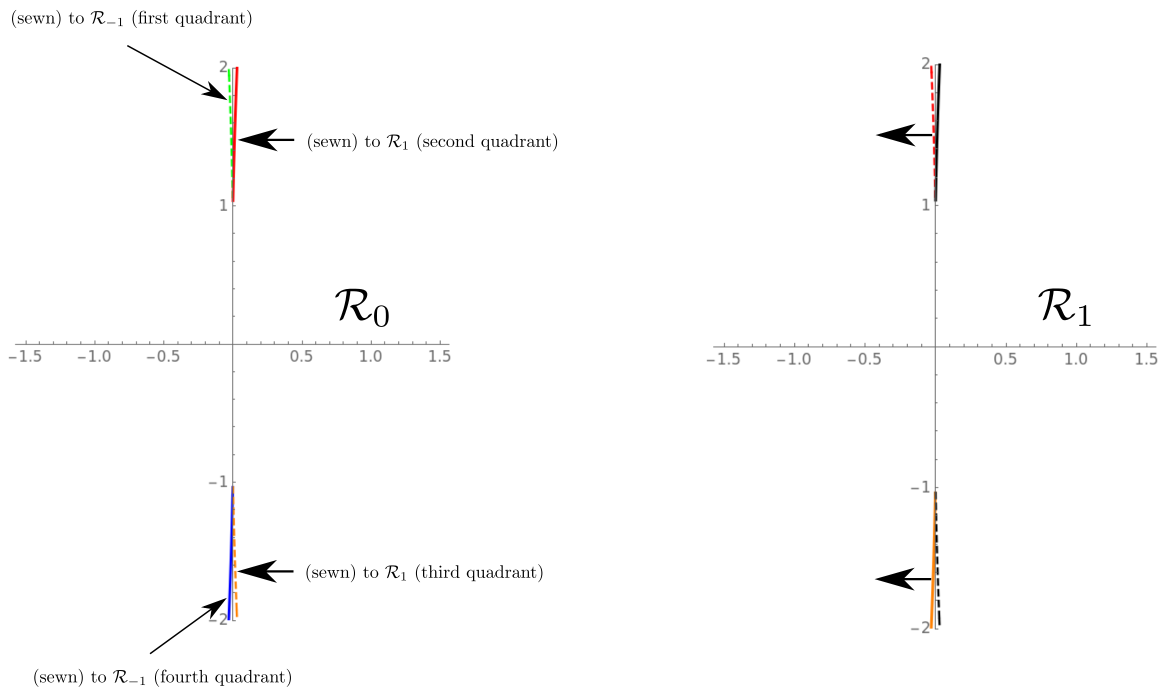

Figure 8: Two fundamental regions $\mathcal{R}_0$ and $\mathcal{R}_1$ in the Riemann surface for the complex tangent function. It would be somewhat better to envision $\mathcal{R}_1$ as a copy of $\mathbb{C}$ (with branch cuts) **above** $\mathcal{R}_0$, but this arrangement does not fit nicely on the page in a figure. One should envision, however, infinitely many fundamental regions $\ldots \mathcal{R}_{-2}$, $\mathcal{R}_{-1}$, $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$, $\ldots$ stacked one above another in a column (and all sewn together).

As something of an aside, this Riemann surface for tangent has the particular property that it can be realized as in Figure 9 (or even more elegantly on the Wikipedia page for "inverse trigonometric functions") as an embedded surface in $\mathbb{R}^3$. Though the same thing is true for the logarithm/exponential, many Riemann surfaces do not have this property. For example, the Riemann surface for the square root function cannot be embedded in $\mathbb{R}^3$. Furthermore, the Riemann surface $\mathcal{R}$ we are considering here is more interesting than the Riemann surface for the logarithm, and $\mathcal{R}$ has a form which has a relatively common application. If you visit the Centergy parking deck at the corner of Williams Street and Ambercrombie (which you can drive around in for ten minutes or so for free) you'll see that it has precisely the form of $\mathcal{R}$. And yet, if you search on the internet for images, the nice one for the tangent function on

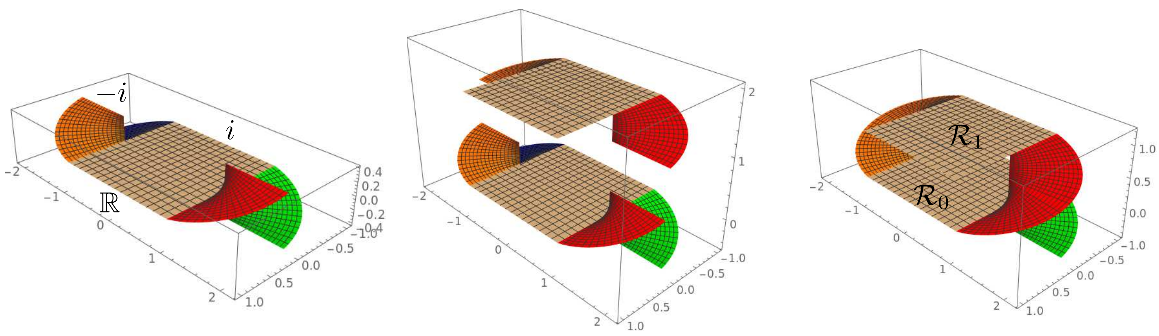

Figure 9: On the left, I have taken a copy of the fundamental region $\mathcal{R}_0$ and twisted up the regions around each edge of the branch cuts that are sewn to $\mathcal{R}_1$. I have also twisted down the regions around the edges of the branch cuts that are sewn to $\mathcal{R}_{-1}$ and colored these twisted regions according to the coloring of the branch cuts in Figure 6. In the middle, I've added a portion of the fundamental region $\mathcal{R}_1$ with twisted branch cuts descending to $\mathcal{R}_0$, and I've colored them with colors corresponding to the sewing. Notice how the fundamental regions are stacked. On the right, I've simply lowered the portion of $\mathcal{R}_1$ to meet $\mathcal{R}_0$ along the ascending branch cuts.

the Wikipedia page is well down the list.

5 Mathematical Software, Proper Implementation, the Complex Arctangent and its restrictions

I hope you have a picture of the Riemann surface

$$\mathcal{R} = \cup_{k \in \mathbb{Z}} \mathcal{R}_k$$

associated with the complex tangent firmly (or more or less firmly) in your mind. (Remember how functions work.) The Riemann surface $\mathcal{R}$ is the natural domain of the complex arctangent:

$$\tan^{-1} : \mathcal{R} \to \mathbb{C}.$$

We can now extend and refine our notion of **branches** of the inverse tangent in a unified manner. Take the **principal arctangent** to be the complex valued function defined on $\mathcal{R}_0$ the central fundamental region of our Riemann surface $\mathcal{R}$. Remember that $\mathcal{R}_0$ is a copy of $\mathbb{C}$ with branch cuts extending from $\pm i$ along the pure imaginary axis. The other **restrictions** of the complex inverse tangent are then given by

$$\tan_k^{-1}(z) = \tan_0^{-1}(z) + \pi k. \tag{10}$$

This is, on the one hand, consistent with our use of $\tan_k^{-1}(x)$ for the values of the branches of the real arctangent. On the other hand, in view of the Riemann surface $\mathcal{R}$ which is the proper domain of the complex arctangent, we see that all branches are just appropriate restrictions and, in particular, are sewn together in a unified manner on $\mathcal{R}$.

A proper implementation of a complex function like the arctangent should have two arguments and look something like $\arctan(z, k)$ corresponding to $\tan_k^{-1}(z)$. This can be really important and useful for you to understand if you're using mathematical software. Unfortunately, the documentation for Mathematica indicates that the Mathematica function `ArcTan[z]` is not a proper implementation, presumably because the complex tangent function is so simple. Even in this case, one should still have the Riemann surface and the corresponding branches in mind. The following question presents itself immediately:

Exercise 8 *If Mathematica does not have a proper implementation of* arctan*, then what does it have?*

Solution: The documentation will not answer this question. One can check (or plot) `ArcTan[`$\cos t + i \sin t$`]` for $t \in \mathbb{R}$ to find that the Mathematica implementation of arctan is precisely what we have called $\tan_0^{-1} : \mathcal{R}_0 \to \mathbb{C}$, the principal complex arctangent.

This is somewhat justified in light of the simple relation (10), but still what Wolfram has can not be called a proper implementation. A similar comment applies to the complex logarithm implemented in Mathematica's `Log[z]`. Not all functions are so simple, and you can find that Mathematica has a proper implementation of the Lambert "W" function given by `ProductLog[k,z]`.

The Mathematica documentation does say that `ArcTan` has a "branch discontinuity" along the branch cuts. In fact, we know the actual complex arctangent has no discontinuity or singularity across those branch cuts. The only actual singularities are at the branch points $\pm i$.

One may finally mention that `ArcTan` does take a second argument in the form `ArcTan[x,y]`. This seems to be primarily intended for use with both x and y real as an alternative for $\mathrm{Arg}(z)$ and returns the argument of the complex number $z = x + iy$.

Now that we understand the inverse tangent with domain the Riemann surface $\mathcal{R}$, let's see if we can use what we know to improve our formula for the complex argument.

6 Conformal Mapping and Problem 9

I guess there is no getting around it: Problem 9 is quite difficult. Finding a formula for the argument is relatively easy as you should observe when you do Problem 10. Finding a formula using arctan is much more difficult. Problem 8 was intended as a hint, and I'm going to use the result of Problem 8 in the discussion of Problem 9 below. This is a little sad because Problem 8 is a really good problem for you to do on your own without having seen my discussion of the answer below. With this in mind, I strongly suggest you complete Problem 8 on your own before you read any further in this section. You can do it! And then we can still be not so sad.

Even with Problem 8 solved as a hint for Problem 9, it is still far from clear how to proceed. The next "hint" would be to go back and embrace and internalize the beauty and power of the Riemann surface structure of the domain $\mathcal{R}$ of the complex inverse tangent

$$\tan^{-1} : \mathcal{R} \to \mathbb{C}.$$

This, I think, is something of a stretch for most of you. And even with the beauty and power of $\mathcal{R}$, Problem 9 is still tough. But let's assume these initial ingredients and see what we can do.

The first thing we might do is go back and consider a bit more carefully the formula

$$\tan^{-1}(y/x).$$

The function $f(z) = y/x = \mathrm{Im}(z)/\mathrm{Re}(z)$ has domain $D = \{z \in \mathbb{C} : \mathrm{Re}(z) \neq 0\}$, and range the **punctured real line** $R = \{x \in \mathbb{R} : x \neq 0\} = \mathbb{R}\backslash\{0\}$. In fact, the function f maps each ray emanating from $0 \in \mathbb{C}$ into D to a real number with the first and third quadrant rays in two-to-one correspondence with the positive real axis and the second and fourth quadrant rays in a two-to-one correspondence with the negative real axis. We can visualize what is happening in this formula as indicated in Figure 10.

Exercise 9 *What is the image of the real axis under the function $f(z) = y/x$?*

Perhaps the first thing to notice in Figure 10 is the collection of rays on the left. All the lines passing through a point in the complex plane is called a **pencil**. Oddly enough, it is called a **pencil of circles** because each straight line in $\mathbb{C}$ is considered to be a circle passing through **the point at infinity**. You can think about each line as a circle with infinite radius and passing through the two points $0 \in \mathbb{C}$ and ∞. We will talk more about this later. It is worth mentioning, however, that whenever one

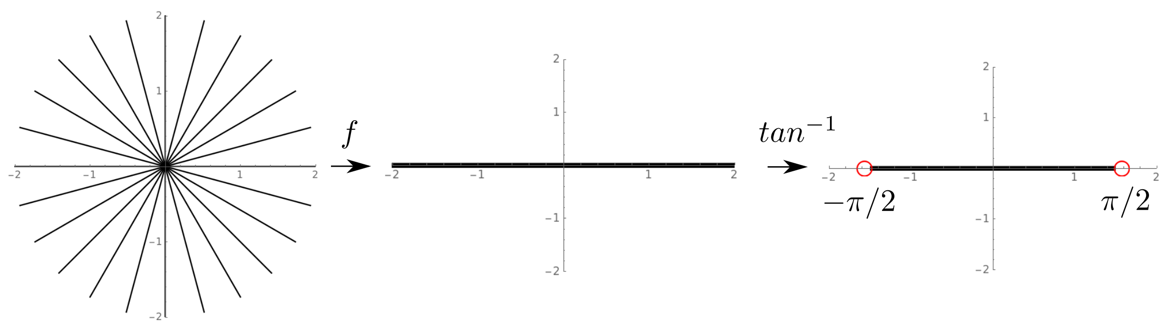

Figure 10: A failed attempt to write down a formula for the argument of a complex number using the inverse tangent. Here, we can think of $f : \mathbb{C}\backslash\{iy : y \in \mathbb{R}\} \to \mathbb{R} \subset \mathbb{C}$ given by $f(z) = y/x$.

has a collection of circles in $\mathbb{C}$ passing through two fixed points, then that is a **pencil of circles**. It should also occur to you that the particular pencil of circles on the left in Figure 10 will have to play a central role in any solution to Problem 9 and also in any formula for the argument of a complex number. Each point in each ray in that picture has the same argument.

Moving on to the middle picture, we have suggestively embedded the real line $\mathbb{R}$ in the complex plane $\mathbb{C}$. The basic idea, naturally, is to replace this simple picture of the range of f with some nontrivial range in $\mathbb{C}$, or more properly, in the proper domain of the complex arctangent, namely the Riemann surface $\mathcal{R}$. This is a first suggestion that we should replace the function f with a complex valued function $g : \mathbb{C} \to \mathcal{R}$.

In order to fill out this idea we can consider the full image of $\tan^{-1} : \mathcal{R} \to \mathbb{C}$. A second suggestion that f should be replaced and about how g might be chosen comes from the fact that $\pi/2$ is an entirely omitted value of $\tan^{-1}$.

Contemplation of Figure 11 suggests attempting to find $g : \mathbb{C} \to \mathcal{R}$ with

$$\operatorname{Arg}(z) = \operatorname{Re}[\tan^{-1}(g(z))]. \tag{11}$$

Notice that if we could arrange for the image of the positive imaginary axis under g to lie along the branch cut in the positive imaginary axis in $\mathcal{R}_0$, then we would have $\operatorname{Re}[\tan^{-1}(g(z))] = \pi/2$. In order to get the arguments for other values of z correct using this approach, we need to ask the question: What is the image under the complex tangent of the **vertical lines** in $\mathbb{C}$? This is Problem 8.

Figure 12 is intended to make several things, more or less, clear. Note first that we have here an additional illustration of the observation mentioned above: If we

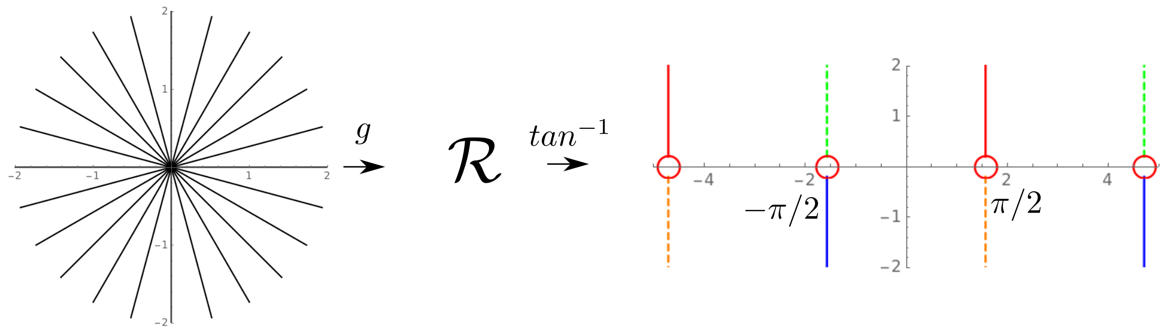

Figure 11: The real number $\pi/2$ can never be the image of any argument of $\tan^{-1}$. The real number $\pi/2$ is not in the range of $\tan^{-1}$. But there are other complex numbers in the range of $\tan^{-1}$ whose real part is $\pi/2$.

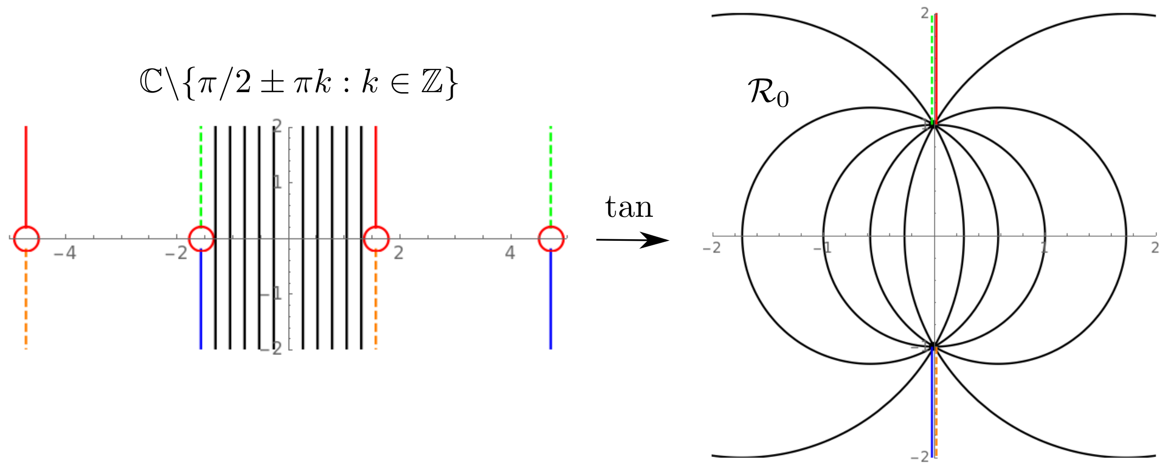

Figure 12: The tangent mapping on vertical lines.

can arrange to have $g(yi) = \eta_1 i$ with $\eta_1 > 1$ and $\eta_1 i$ on the right edge of the branch cut in $\mathcal{R}_0 \subset \mathcal{R}$, then $\tan^{-1}(g(yi)) \in \{\psi \in \mathbb{C} : \operatorname{Re}\psi = \pi/2\}$. Therefore, we would have $\operatorname{Arg}(yi) = \operatorname{Re}[\tan^{-1}(g(yi))]$, which is what we want. Viewed from the reverse direction, the part of the vertical line with real part $\pi/2$ and imaginary part positive maps by the tangent onto the right edge of the upper branch cut in $\mathcal{R}_0$.

The other vertical lines on the left in Figure 12 map to circles passing through $\pm i$. These circles form a pencil of circles (of course). Each portion of a circle in this pencil of circles between $-i$ and $+i$ maps by the inverse tangent to a vertical line with constant real part. Thus, we would like to map the pencil of circles on the left in Figure 10 and Figure 11 by some function g, in a more or less one-to-one way, onto the pencil of circles on the right in Figure 12. If we can do this in the right way, then presumably something along the lines of (11) will be correct. Before I describe such a map, let me note something else about Figure 12 and Problem 8. The vertical lines on the left in Figure 12 are **not** a pencil of circles. These "circles" only intersect in one point, the point at ∞. So the tangent map gives a correspondence between a collection of circles which is not a pencil and a pencil of circles. That's sort of interesting. In complex analysis, most nice functions preserve angles, i.e., are conformal, and some conformal functions preserve circles in the sense that circles map to circles (and straight lines). The most well-know class of conformal mappings taking circles to circles are called **linear fractional transformations** and have the form

$$h(z) = \frac{az+b}{cz+d}$$

where a, b, c, and d are complex numbers.

Our discussion so far should suggest, at least roughly speaking, that we want to find a function $g : \mathbb{C} \to \mathbb{C}$ taking the pencil of circles on the left in Figure 10 and Figure 11 onto the pencil of circles in $\mathcal{R}$ indicated on the right in Figure 12. We will also need the real axis (with argument $\theta = 0$) to map onto the pure imaginary axis between $-i$ and i. This suggests something like the illustration in Figure 13. This means the mapping g will not preserve angles at the origin. In particular, it looks like we want the image of the angle determined by the real and imaginary axes to have an angle of π (or 180 degees) at the image point i. Noting that a complex square has this property, the mapping I will suggest is given by

$$g(z) = i\,\frac{1-z^2}{1+z^2}. \tag{12}$$

Let's discuss this function as a composition of three useful complex functions. First there is the square $g_1(z) = z^2$ which we can use to map our initial pencil into the Riemann surface for z^2, which you can think of as a double cover of the complex plane with a branch point of order two at the origin as indicatedin Figure 14. When working through how the square function works, it is useful to use the polar form for complex numbers to verify that the argument is doubled under the square:

Exercise 10 *Use complex multiplication to verify that*

$$\left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

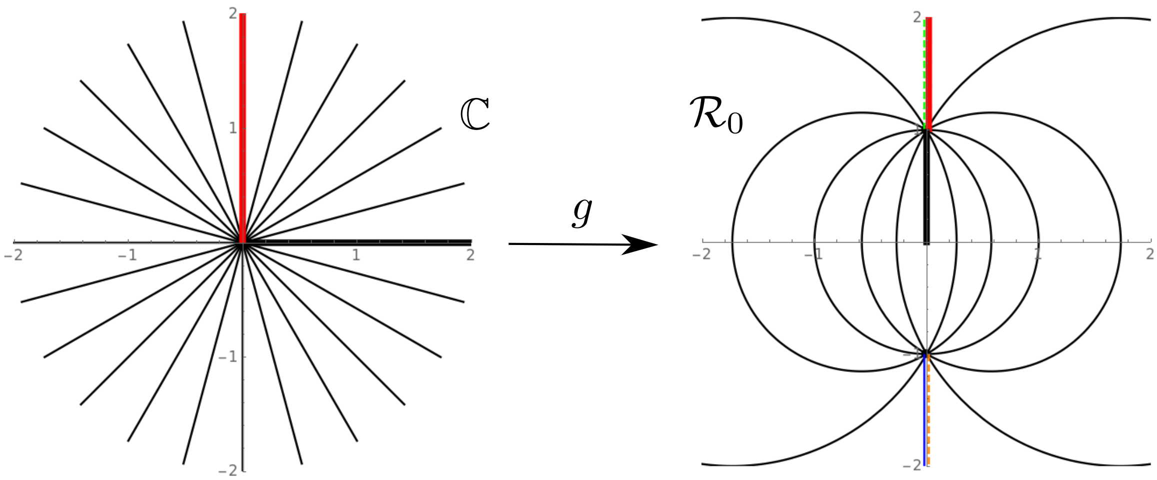

Figure 13: Correspondence between two pencils of circles.

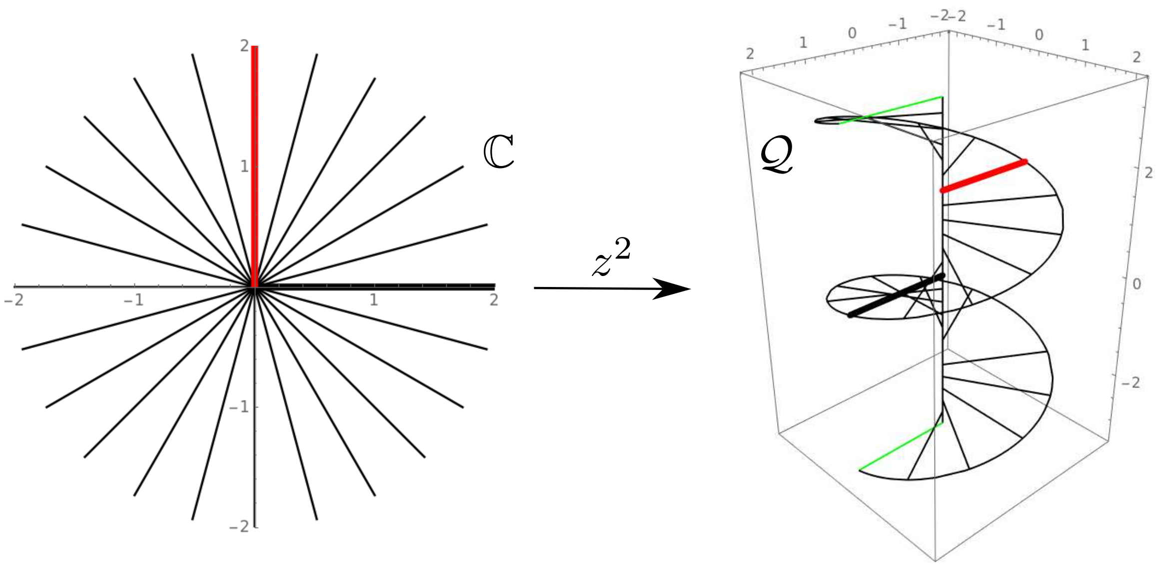

Figure 14: The pencil through 0 and ∞ mapped by z^2 into the Riemann surface $\mathcal{Q}$ for z^2.

so that in particular,

$$\mathrm{Arg}(z^2) = 2\,\mathrm{Arg}(z).$$

Thus, our pencil of circles is pulled apart using z^2, but the center is still at the origin, which is a branch point in $\mathcal{Q}$. Now, we will attempt to send the origin 0 in $\mathcal{Q}$ to i and ∞ to $-i$ using a linear fractional transformation (LFT). The LFT given by

$$h(\zeta) = \frac{1-\zeta}{1+\zeta}$$

sends $0 \in \mathbb{C}$ to 1 and ∞ to -1 when considered as a conformal map on the extended complex plane $\mathbb{C} \cup \{\infty\}$, so a rotation by $\pi/2$ counterclockwise provides essentially what we want as indicated in Figure 15.

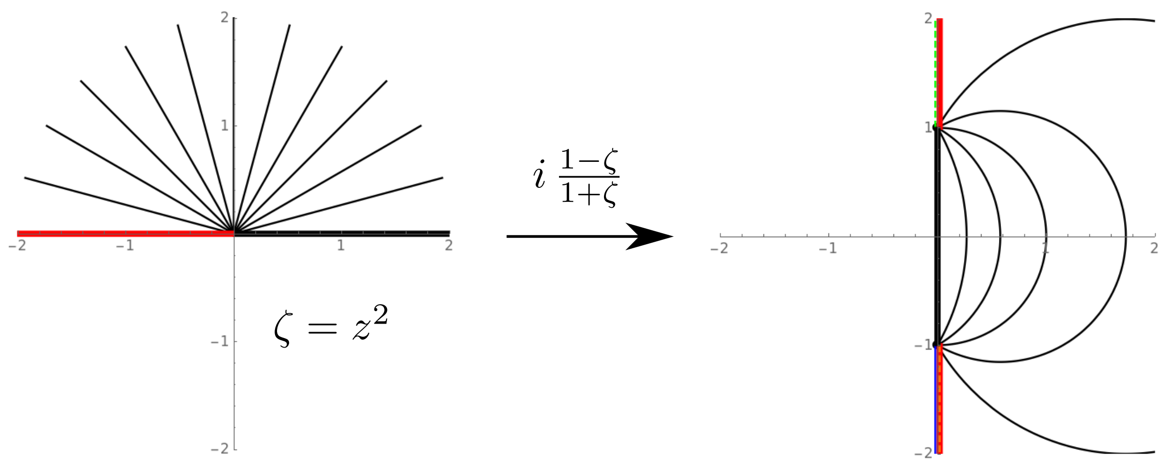

Figure 15: Linear fractional transformations preserve circles. Here we use the mapping $\zeta \mapsto i(1-\zeta)/(1+\zeta)$ to map $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$.

Exercise 11 *Find a linear fractional transformation determined by the following values:*

$$i \mapsto i$$
$$-i \mapsto -i$$
$$0 \mapsto 1$$
$$1 \mapsto \infty.$$

What is the image of ∞ under your transformation? Notation: When we write $a \mapsto b$ we mean that we have a function, say, $f : A \to B$, and we have $a \in A$ and $b \in B$ with $f(a) = b$. So $a \mapsto b$ is read a "maps to" b or "b is the image of a" (under whatever function we are talking about).

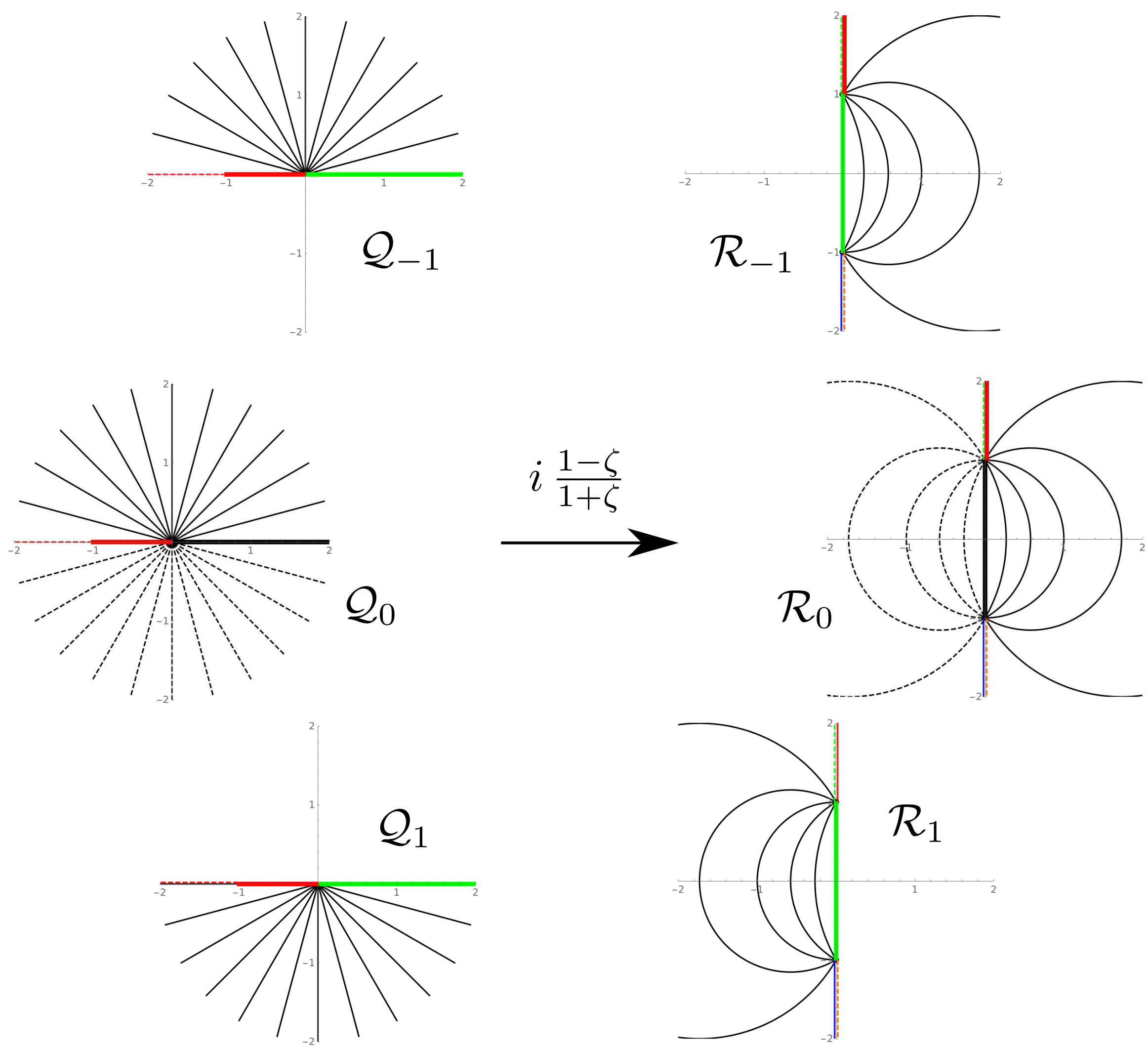

Figure 16: Here we use the mapping $\zeta \mapsto i(1-\zeta)/(1+\zeta)$ to map $\mathcal{Q} \to \mathcal{R}$. It may be helpful to realize that the three sheets on the left representing domains in $\mathcal{Q}$ are visualized in a different way on the right in Figure 14; there are actually only two sheets for all of the Riemann surface $\mathcal{Q}$ as the square function has a branch point of order two.

If we want to use h to map from $\mathbb{C}$ to $\mathcal{R}$ (or even $h : \mathcal{Q} \to \mathcal{R}$), then we see that part of the negative real axis (thick red line) has already spilled out of $\mathcal{R}_0$. The rays extending into the fourth quadrant (with negative argument) are happily mapped into $\mathcal{R}_0$. An attempt to show the full visualization (up to some values along branch cuts) is shown in Figure 16.

Combining the various elements of our discussion above we arrive at the following (improved) formula for the argument of a complex number:

$$\operatorname{Arg}(z) = \begin{cases} \operatorname{Re}\left[\tan_{-1}^{-1}\left(i\,\dfrac{1-z^2}{1+z^2}\right)\right], & \operatorname{Re} z \le 0, \operatorname{Im} z < 0, \\[2ex] \operatorname{Re}\left[\tan_{0}^{-1}\left(i\,\dfrac{1-z^2}{1+z^2}\right)\right], & \operatorname{Re} z \ge 0 \\[2ex] \operatorname{Re}\left[\tan_{1}^{-1}\left(i\,\dfrac{1-z^2}{1+z^2}\right)\right], & \operatorname{Re} z \le 0, \operatorname{Im} z \ge 0. \end{cases}$$

Some of what I've written down here for these cases is not entirely correct. First of all, I've been a bit sloppy about the edges of the branch cuts. For example, the middle case should really be restricted by

$$\operatorname{Re} z > 0, \quad \text{or } \operatorname{Re} z = 0,\ \operatorname{Im} z > 0, \text{ and } z \ne i.$$

If $\operatorname{Im} z < 0$ (and $\operatorname{Re} z = 0$ so that z is purely imaginary on the negative imaginary axis), then the square will fall into $\mathcal{Q}_{-1}$, the image under the linear fractional transformation will fall into $\mathcal{R}_{-1}$, and we should really use the branch $\tan_{-1}^{-1}$ of the complex arctangent instead of $\tan_0^{-1}$.

You should see another (more serious) problem. I still don't have a unified formula for $\operatorname{Arg}(-i)$ and $\operatorname{Arg}(i)$. These two values with $\operatorname{Arg}(\pm i) = \pm\pi/2$ should still be included as another case. Using the mapping g given above, this omission of $\pm i$ is unavoidable. It arises because tan has a singularity at $\pi/2+2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$, and with the mapping g, based on a linear fractional transformation, something must go to ∞ corresponding to the singularity. And indeed you see $g(\pm i) = \infty$.

Aside from these shortcomings, we have a formula (modulo understanding the complex structure of the tangent function) representing a vast improvement over what we could handle with the real arctangent. Notice that each of the cases in our formula above has essentially exactly the same form; you just need to use the correct branch of the complex arctangent. With the real arctangent we had to omit the entire imaginary axis. Consider the following example:

Let $z = 2i$. Then

$$\operatorname{Re}\left[\tan_0^{-1}\left(i\frac{1-(2i)^2}{1+(2i)^2}\right)\right] = \operatorname{Re}\left[\tan_0^{-1}\left(\frac{-5i}{3}\right)\right] = \operatorname{Re}\left[\frac{\pi}{2} - i\coth^{-1}\left(\frac{5}{3}\right)\right] = \pi/2.$$

So for every other purely imaginary number (other than $-i$ and i) our formula works perfectly. It also distinguishes seamlessly between the first and third quadrants (and second and fourth quadrants) not by ad hoc cases but using the natural branching structure of the complex tangent.

7 Shortcomings and Confessions

We have been successful in using the complex arctangent $\tan^{-1} : \mathcal{R} \to \mathbb{C}$ to write down a formula for $\operatorname{Arg}(z)$ when $z \in \mathbb{C}\backslash\{\pm i\}$. The two remaining points $\pm i$ are inherently excluded from the formula, as mentioned above, due to the singularity at

$$\pi/2 + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots$$

in the (complex) tangent function. There may be another choice of $g : \mathbb{C} \to \mathcal{R}$ which avoids this shortcoming.

The choice of g we have used, however, is the "obvious" one. At least it is the most natural one. After the discussion above, you may be a little incredulous at these assertions. You may ask

1. Why is it natural?
2. And, setting the notion of "obviousness" aside, how did you (magically) come up with this formula
$$g(z) = i\,\frac{1-z^2}{1+z^2}\ ? \tag{13}$$

Of course, I tried to motivate the formula in (13) by using pencils of circles and some things I know about linear fractional transformations (which maybe you now know too). But really, the answer to both of these questions is the following:

In Problem 10 you should have convinced yourself that the Riemann surface

$$\mathcal{L} = \cup_{k=-\infty}^{\infty}\mathbb{C}_k$$

associated with the complex exponential $z \mapsto e^z$ is one with a single branch point of infinite order at the origin. As a result, there are infintely many branches of the complex logarithm generated from

$$\log_0(z) = \log|z| + i\,\mathrm{Arg}(z),$$

for $z \in \mathbb{C}_0$ with $|z| > 0$ and $-\pi < \mathrm{Arg}(z) \le \pi$, and

$$\log_k : \mathbb{C}_k \to \mathbb{C} \qquad \text{by} \qquad \log_k(z) = \log_0(z) + 2\pi k\ i.$$

In particular, from the point of view of complex analysis (or someone who understands the complex logarithm) the obvious formula for $\mathrm{Arg}(z)$ is simply

$$\mathrm{Arg}(z) = \mathrm{Im}[\log(z)]. \tag{14}$$

This means, in particular, that if we are looking for a formula of the form

$$\mathrm{Arg}(z) = \mathrm{Re}\left[\tan^{-1} g(z)\right],$$

based on the idea that the image of the pencil of circles on the left in Figure 10 and Figure 11 under $\psi(z) = \tan^{-1} g(z)$ will be the family of vertical lines shown on the left in Figure 12, then we already know an "obvious" function which does this. Namely, if we rotate the image of $\log z$ clockwise by $\pi/2$, then the horizontal lines according to which

$$\mathrm{Arg}(z) = \mathrm{Im}(\log z)$$

become the vertical lines with $\mathrm{Arg}(z) = \mathrm{Re}[\psi(z)]$. Of course, this is where we pick up the singularities at $\pi/2 + k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$. Nevertheless, this means we should want (and write down)

$$\psi(z) = \tan^{-1}(g(z)) - = i \log z.$$

Exercise 12 *Simplify*

$$g(z) = \tan(-i \log z).$$

Hint:

$$\tan w = -i\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}.$$

Assignment Problems

Problem 1 *The* **complex conjugate** *of a complex number* $z = x + iy$ *is* $\bar{z} = x - iy$. *Find* $z + \bar{z}$, $z - \bar{z}$, *and* $z\bar{z}$. *Express* $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, *and* $|z|$ *in terms of* z *and* $\bar{z}$.

Problem 2 *Draw the following subsets in the complex plane.*

(a) *(Boas 2.4)* $\{z \in \mathbb{C} : |z - 2| = 1\}$.

(b) *(Boas 2.5.46)* $\{z \in \mathbb{C} : z = -i\bar{z}\}$.

(b) *(Boas 2.5.52)* $\{z \in \mathbb{C} : \mathrm{Re}\, z = 1\}$.

Problem 3 *Check that the series expansions*

$$\cos z = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} z^{2\ell}}{(2\ell)!} \qquad \text{and} \qquad \sin z = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} z^{2\ell+1}}{(2\ell+1)!}$$

hold for $z \in \mathbb{C}$ *using the definitions*

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

given in (5) and the series expansion (definition) for the exponential.

Problem 4 *Show that if one defines cosine and sine for complex arguments using the series expansions*

$$\cos z = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} z^{2\ell}}{(2\ell)!} \qquad \text{and} \qquad \sin z = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} z^{2\ell+1}}{(2\ell+1)!},$$

then the formulas

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

given in (5) still hold.

Problem 5 *Check that for* $z = x + iy$

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

using the definitions

$$\cosh x = \frac{e^{x} + e^{-x}}{2} \qquad \text{and} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

for the real hyperbolic cosine and sine.

Problem 6 *Plot the real hyperbolic cosine and sine. Derive the real McClauren series expansions for* $\cosh x$ *and* $\sinh x$ *and determine the radii of convergence for these series.*

Problem 7 *Check, as claimed above, that the image of the horizontal segment* $\{x + iy : -\pi/2 < x < \pi/2\}$ *(for* y *fixed) lies on a circle with center at* $i \coth(2y)$ *on the imaginary axis.*

Problem 8 *Determine the image of a vertical line* $\{x + iy \in \mathbb{C} : y \in \mathbb{R}\}$ *(with* x *fixed) under the complex tangent function.*

Problem 9 *Express* $\mathrm{Arg}(z)$ *properly in terms of a branch of the complex inverse tangent function.*

Problem 10 *Determine the value of the complex logarithm and determine the associated Riemann surface by understanding the complex exponential as a mapping.*