Exam 1
Due Friday September 18, 2020

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Problem 1 *Find the power series expansion and radius of convergence with respect to the center of expansion $z = 0$ for the following complex functions:*

(a) *(Boas 14.2.36)* $\sqrt{1+z^2}$.

(b) *(Boas 14.2.39)* $z/(z^2+9)$.

Problem 2 *(Boas 14.2.45) If $f(z) = u + iv$ is a complex differentiable function and $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ is the real vector field given by $\mathbf{v} = (v, u)$, then show*

(a) $\operatorname{div} \mathbf{v} = 0$.

(b) $\operatorname{curl} \mathbf{v} = 0$.

Problem 3 *(Boas 14.2.46) Find a system of Cauchy-Riemann equations in polar coordinates for the real and imaginary parts $\xi = \xi(r, \theta)$ and $\eta = \eta(r, \theta)$ of a complex differentiable function $f = \xi + i\eta$. Hint: Write $x = r\cos\theta$ and $y = r\sin\theta$ and apply the chain rule to $u = u(x, y)$ and $v = v(x, y)$.*

Problem 4 *(Boas 14.2.48) Express the real and imaginary parts ξ and η of $f(z) = \sqrt{z}$ in polar coordinates, and verify that the Cauchy-Riemann equations in polar coordinates derived in the previous problem hold for ξ and η.*

Problem 5 *(Boas 14.2.47) Find Laplace's equation in polar coordinates.*

Problem 6 *(Boas 14.2.55) Show that $u(x, y) = 3x^2y - y^3$ is harmonic and find a harmonic conjugate for u.*

Problem 7 *Compute*

$$\int_{\partial B_r(z_0)} \frac{1}{z - z_0}.$$

Make the calculation from scratch; do not use the Cauchy integral formula or the residue theorem. What can you say about

$$\lim_{r \searrow 0} \left| \int_{\partial B_r(z_0)} \frac{f(z)}{z - z_0} - \int_{\partial B_r(z_0)} \frac{f(z_0)}{z - z_0} \right| ?$$

Hint: Remember $|\int q| \leq \int |q|$.

Problem 8 *(Boas 14.3.21) Differentiate the Cauchy integral formula repeatedly (under the integral sign) to obtain a formula for the n-th derivative of a complex differentiable function f.*

Problem 9 *(Boas 14.11.3) Prove Liouville's Theorem: A bounded entire function is constant. The word* **entire** *here means* complex differentiable on all of $\mathbb{C}$. *Hint: Use the previous problem with $n = 1$ and $\Gamma = \partial B_r(z)$, then estimate the integral.*

Problem 10 **(a)** *Use mathematical software to plot the image of the composition $g \circ \gamma(t)$ in the complex plane where $g(z) = e^z$ and $\gamma(t) = re^{it}$ for $0 \leq t \leq 2\pi$ parameterizes a semicircle for various values of r between* 0.1 *and* 3.

(b) *Consider the function $f : \mathbb{C} \backslash \{\pm i\} \to \mathbb{C}$ by*

$$f(z) = \frac{e^{iz}}{z^2 + 1}$$

which has a simple pole at $z = i$. Integrate around $\partial B_R^+(0) = \partial\{z \in \mathbb{C} : |z| < R \text{ and } \operatorname{Im}(z) > 0\}$ and apply the residue theorem to calculate the real integral

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2 + 1}\, dt.$$

(Bonus) At the end of my posted solution for Assignment 2 Problem 1, I mentioned several guesses (or conjectures) concerning the values of the alternating harmonic series

$$f(z) = \sum_{j=0}^{\infty} (-1)^j \frac{z^{j+1}}{j+1}.$$

I wrote there "I have no idea how to prove these assertions."

Find the function f explicitly.

Hints:

(a) Think carefully about Exercise 7 in my solution to Problem 1 of Assignment 2.

(b) Review Chapter 2 Sections 6 and 7 of Boas.

(c) Use (your solution of) Problem 10 of Assignment 1 to see geometrically precisely what is going on and get the formula.

When you are done you should be able to produce/reproduce Figure 4 of my solution to Problem 1 of Assignment 2 with relative ease.