Math 6701, Exam 1: (practice) Name/Section: ________________

1. (20 points) (2.4.10) Find the inverse matrix A^{-1} if

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}.$$

Solution: We append the identity matrix and row reduce:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix}.$$

Thus, we see the inverse is given by

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}.$$

Name and section: ____________________

2. (20 points) (3.2.18) Let

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 7 \\ 10 \end{pmatrix} \right\}.$$

Determine if S is linearly independent. Explain your reasoning.

Solution: We assemble the vectors into a 4×3 matrix and apply the row reduction algorithm to obtain

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \\ 1 & 4 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \\ 0 & 3 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From this we see there are two pivots, the rank is 2, and hence, the original collection contains only a subset of two linearly independent vectors. So the answer to the question is that the set S is linearly dependent and not linearly independent. We can proceed one more step in the row reduction to find that the third vector is an explicit linear combination of the first two as follows:

$$(1, 4, 7, 10)^T = -2(1, 1, 1, 1)^T + 3(1, 2, 3, 4)^T.$$

Name and section: ____________________

3. (20 points) (4.3.16) Find the matrix of

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

with respect to the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Solution: Note that

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, we need to express the images in terms of the given basis vectors and record the coefficients in the columns:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Name and section: ____________________

4. (20 points) (7.2.9) Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{pmatrix}.$$

Solution:

$$\det \begin{pmatrix} 3-\lambda & -2 & 5 \\ 1 & -\lambda & 7 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)(\lambda^2 - 3\lambda + 2) = -(\lambda-2)^2(\lambda-1).$$

Thus, the eigenvalues are 1 (with multiplicity 1) and 2 (with multiplicity 2).

Name and section: ____________________

5. (20 points) (7.4.17) Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution: We first find the eigenvalues and eigenvectors.

$$\det \begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)[(1-\lambda)^2-1]-(1-\lambda-1)+[1-(1-\lambda)] = -\lambda^2(\lambda-3).$$

Solving $A\mathbf{v} = 0$ for the eigenvectors corresponding to $\lambda = 0$, we find a two dimensional eigenspace of vectors

$$\mathbf{v} = \begin{pmatrix} -t_2 - t_3 \\ t_2 \\ t_3 \end{pmatrix} = t_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Solving $(A - 3I)\mathbf{v} = 0$ for the eigenvectors corresponding to $\lambda = 3$, we find the one-dimensional subspace spanned by $(1, 1, 1)^T$. Thus, we take the matrix of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ in the basis

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

That matrix is

$$\begin{aligned} \Lambda &= \frac{1}{3} \begin{pmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \end{aligned}$$