Math 6701, Final Exam Spring 2015 Name and section: ____________

1. (20 points) (3.11.18) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying $L(\mathbf{e}_1) = (-1,1,3)$, $L(\mathbf{e}_2) = (1,2,0)$, and $L(\mathbf{e}_3) = (3,0,2)$ where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2\}$ is the standard basis for $\mathbb{R}^3$. Diagonalize the linear transformation L.

Solution: I will first switch the notation of $\mathbb{R}^3$ to columns, so the matrix associated to the linear transformation becomes

$$A = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix}.$$

Next, I will compute $\det(A - \lambda I)$ to find the eigenvalues. Expanding along the last column of $A - \lambda I$, we find

$$3(-3)(2-\lambda)+(2-\lambda)[(-1-\lambda)(2-\lambda)-1] = (2-\lambda)[-9+\lambda^2-\lambda-3] = (2-\lambda)(\lambda+3)(\lambda-4).$$

For $\lambda = -3$, reducing $A - \lambda I$ we find

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & 0 \\ 3 & 0 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & 0 \\ 0 & 9 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $v_2(-5,1,3)^T$ is a solution of the associated homogeneous system for any v_3, and $(-5,1,3)^T$ is an eigenvector.

For $\lambda = 2$, reducing $A - \lambda I$ we find

$$\begin{pmatrix} -3 & 1 & 3 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $v_3(0,-3,1)^T$ is a solution of the associated homogeneous system for any v_3, and $(0,-3,1)^T$ is an eigenvector.

For $\lambda = 4$, reducing $A - \lambda I$ we find

$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -2 & 0 \\ 3 & 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 0 \\ 0 & 6 & -2 \\ 0 & 9 & -3 \end{pmatrix}.$$

Thus, $v_2(2,1,3)^T$ is a solution of the associated homogeneous system for any v_3, and $(2,1,3)^T$ is an eigenvector.

Since we have a basis of real eigenvectors, we can diagonalize. Setting

$$Q^{-1} = \begin{pmatrix} -5 & 0 & 2 \\ 1 & -3 & 1 \\ 3 & 1 & 3 \end{pmatrix},$$

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we find

$$Q = \frac{1}{70}\begin{pmatrix} -10 & 2 & -6 \\ 0 & -21 & 7 \\ 10 & 5 & 15 \end{pmatrix},$$

and

$$QAQ^{-1} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

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2. (20 points) Solve the system of ordinary differential equations

$$\begin{cases} x' = -x + y + 3z \\ y' = x + 2y \\ z' = 3x + 2z. \end{cases}$$

Solution: Notice that this system can be written as

$$\mathbf{x}' = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 2 & 0 \\ 3 & 0 & 2 \end{pmatrix} \mathbf{x}.$$

In view of the previous problem, the general solution of this system is therefore

$$ae^{-3t}\begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix} + be^{2t}\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + ce^{4t}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

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3. (20 points) (12.11.6) Use the method of Frobenius to find two linearly independent solutions of the ODE

$$3xy'' + (3x+1)y' + y = 0.$$

Solution: Substituting

$$y = \sum_{j=0}^{\infty} a_j x^{j+\beta}$$

into the equation we find

$$\sum_{j=0}^{\infty}[3(j+\beta-1)(j+\beta)+(j+\beta)]a_j x^{j+\beta-1} + \sum_{j=0}^{\infty}[3(j+\beta)+1]a_j x^{j+\beta} = 0. \qquad (1)$$

The $x^{\beta-1}$ term gives the indicial equation

$$3\beta^2 - 2\beta = \beta(3\beta-2) = 0$$

corresponding to $a_0 \neq 0$. Notice we have the first case of Frobenius' theorem in which there are two distinct roots, and their difference is not an integer. In fact, one root is $\beta = 0$, so we get a standard power series for one solution.

Taking $\beta = 0$ we find for $j \geq 1$

$$[3j(j-1)+j]a_j + [3(j-1)+1]a_{j-1} = j(3j-2)a_j + (3j-2)a_{j-1} = 0$$

or

$$a_j = -\frac{1}{j}a_{j-1}.$$

It follows that $a_j = (-1)^j a_0/j!$ for all j, and a first solution is $y_1(x) = e^{-x}$.

For the second solution, we take $\beta = 2/3$. Replacing the a_j's with b_j's and putting $\beta = 2/3$ in (1) we find that for $j \geq 1$

$$[3(j-1/3)(j+2/3)+(j+2/3)]b_j + [3(j-1/3)+1]b_{j-1} = j(3j+2)b_j + 3jb_{j-1} = 0$$

or

$$b_j = -\frac{3}{3j+2}a_{j-1}.$$

In particular,

$$b_1 = -\frac{3}{5}b_0; \quad b_2 = \frac{3^2}{(8)(5)}b_0; \quad b_3 = \frac{3^3}{(11)(8)(5)}b_0,$$

and a second solution is given by

$$\begin{aligned} y_2(x) &= x^{2/3}\left(1 + \sum_{j=1}^{\infty}\frac{(-3)^j}{(3j+2)\cdots(5)}x^j\right) \\ &= x^{2/3} + \sum_{j=1}^{\infty}\frac{(-3)^j}{(3j+2)\cdots(5)}x^{j+2/3}. \end{aligned}$$

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4. (20 points) Find a conformal mapping ϕ of the open unit disk onto the 45 degree wedge $\{z = a + bi : 0 < Arg(z) < \pi/4\}$.

Solution: If we can map the disk onto the upper half plane, then we can use a branch of the 1/4 root to map the upper half plane onto the wedge.

We know there is a linear fractional transformation

$$g(z) = \frac{az+b}{cz+d}$$

mapping the disk onto the upper half plane with $\pm 1 \to \pm 1$, $0 \mapsto i$, $i \to \infty$ and $-i \to 0$. Using any combination of these facts, we deduce that this mapping is

$$g(x) = -i\frac{z+i}{z-i} = \frac{1-iz}{z-i}.$$

Now, taking a branch of $z^{1/4}$ defined for example on $\{z : -\pi < Arg(z) < \pi\}$ by

$$z \mapsto |z|^{1/4}[\cos(\text{Arg}(z)/4) + i\sin(\text{Arg}(z)/4)],$$

the desired mapping is $\phi(z) = g(z)^{1/4}$.

Note, if we wish to be very explicit, we may write $z = x + iy$ and plug into the formula for g to find

$$g(z) = \frac{(1+y) - xi}{x + (y-1)i} = \frac{(1+y) - xi}{x - (1-y)i} = \frac{2x + (1 - y^2 - x^2)i}{x^2 + (1-y)^2}.$$

In particular,

$$|g(z)| = \frac{\sqrt{1 + 2(x^2 - y^2) + (x^2 + y^2)^2}}{x^2 + (1-y)^2}.$$

and

$$\text{Arg}(z) = \cos^{-1}\left(\frac{2x}{\sqrt{1 + 2(x^2 - y^2) + (x^2 + y^2)^2}}\right).$$

Notice the second formula gives $\text{Arg}(z)$ between 0 and π for all $z = x + iy$ with $y \geq 0$. Thus,

$$\begin{aligned}
\phi(z) &= \phi(x+iy) \\
&= \left(\frac{\sqrt{1 + 2(x^2 - y^2) + (x^2 + y^2)^2}}{x^2 + (1-y)^2}\right)^{1/4} \left[\cos\left(\frac{1}{4}\cos^{-1}\left(\frac{2x}{\sqrt{1 + 2(x^2 - y^2) + (x^2 + y^2)^2}}\right)\right)\right. \\
&\qquad \left. \sin\left(\frac{1}{4}\cos^{-1}\left(\frac{2x}{\sqrt{1 + 2(x^2 - y^2) + (x^2 + y^2)^2}}\right)\right)\right].
\end{aligned}$$

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5. (20 points) Use the residue theorem to determine

$$\int_{-\infty}^{\infty} \frac{1}{1+t^6}\, dt.$$

Hint: Integrate around the boundary of a half disk of large radius r.

Solution: We let $\gamma(t) = t$ for $-r \le t \le r$ and $\beta(t) = re^{it}$ for $0 \le t \le \pi$. The concatenation Γ of γ and β is the boundary of a half disk. And we consider

$$\int_\Gamma \frac{1}{1-z^6}.$$

The integrand has six simple poles at θ, i, θ^5, $-\theta^5$, $-i$ and $-\theta^5$ where

$$\theta = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

is the principle sixth root of -1. Three of these roots (the first three) have positive imaginary part and lie in the half disk for $r > 1$. Thus, the residue theorem tells us

$$\int_\Gamma \frac{1}{1+z^6} = 2\pi i \left[\operatorname{Res}_{z=\theta} \frac{1}{1+z^6} + \operatorname{Res}_{z=i} \frac{1}{1+z^6} + \operatorname{Res}_{z=\theta^5} \frac{1}{1+z^6} \right].$$

On the other hand,

$$\int_\Gamma \frac{1}{1+z^6} = \int_{-r}^{r} \frac{1}{1+t^6}\, dt + \int_0^\pi \frac{rie^{it}}{1+r^6 e^{6it}}\, dt.$$

Noting that each of these integrals has a finite limit as $r \to \infty$ with the integral around the circular boundary vanishing in the limit, we see that we only need to calculate the residues to determine the value of the desired integral.

Notice that $1 + z^6 = (1 + iz^3)(1 - iz^3)$ with the cube roots of i (which are θ, θ^5 and $-i$) being roots of the first factor. In particular,

$$1 + iz^3 = i(z-\theta)(z-\theta^5)(z+i).$$

Thus,

$$\operatorname{Res}_{z=\theta} \frac{1}{1+z^6} = \frac{z-\theta}{1+z^6}\Big|_{z=\theta} = -\frac{i}{(\theta-\theta^5)(\theta+i)(1-i\theta^3)}$$

Since $\theta^5 = -\sqrt{3}/2 + i/2$ and $\theta^3 = i$, we have

$$\operatorname{Res}_{z=\theta} \frac{1}{1+z^6} = -\frac{i}{(\sqrt{3})(\sqrt{3}/2 + 3i/2)(2)} = -\frac{\sqrt{3}+i}{12}.$$

Similarly,

$$\mathrm{Res}_{z=\theta^5}\frac{1}{1+z^6} = -\frac{i}{(\theta^5-\theta)(\theta^5+i)(1-i\theta^{15})} = \frac{\sqrt{3}-i}{12}.$$

Finally, i is a cube root of $-i$ and is a root of the second factor which we can write (using synthetic division for example) as $(1-iz^3) = (z-i)(-iz^2+z+i)$. It follows that

$$\mathrm{Res}_{z=i}\frac{1}{1+z^6} = \frac{1}{(1+i^4)(-i^3+i+i)} = \frac{1}{(2)(3i)} = -\frac{i}{6}.$$

Summing the residue (and multiplying by $2\pi i$) we find

$$\int_\Gamma \frac{1}{1+z^6} = 2\pi i\left[-\frac{i}{6}-\frac{i}{6}\right] = \frac{2\pi}{3}.$$