Final Exam on ODEs, and a little linear algebra and complex analysis Due Monday December, 2020

John McCuan

December 3, 2020

You know (roughly speaking) what it means for u to be a weak solution of the FTC equation

$$u' = f. \tag{1}$$

More precisely, given $f \in L^1_{loc}(a,b)$, a function $u \in W^1(a,b)$ is a weak solution of $u' = f$ if f is the weak derivative of u, that is,

$$-\int u\phi' = \int f\phi \qquad \text{for every } \phi \in C_c^\infty(a,b)$$

where all integrals are over the interval (a,b). You also know that $W^1(a,b)$ is nothing more than the subset of functions in $L^1_{loc}(a,b)$ having a weak derivative in $L^1_{loc}(a,b)$. What you do not really know so well (presumably) is what exactly is the space of functions $L^1_{loc}(a,b)$. You should, however, know the following about $L^1_{loc}(a,b)$:

1. These are the **locally integrable functions** $u : (a,b) \to \mathbb{R}$.

2. $L^1_{loc}(a,b)$ contains $C^0(a,b)$, including functions, like $u : (a,b) \to \mathbb{R}$ by $u(x) = 1/(x-a)$, which are not globally integrable.

3. $L^1_{loc}(a,b)$ contains many discontinuous functions like $\chi_{\mathbb{Q}} : (a,b) \to \mathbb{R}$ by $\chi_{\mathbb{Q}}(x) = 1$ if $x \in \mathbb{Q}$ and $\chi_{\mathbb{Q}}(x) = 0$ if $x \notin \mathbb{Q}$.

Note carefully the the last item here. You may not know all the functions in $L^1_{loc}(a,b)$, but you should know that these functions can be very far from having a classical derivative. You should have in your mind that they can be very irregular functions.

Now a function in $L^1_{loc}(a,b)$ with a weak derivative, that is a function in $W^1(a,b)$ **does** have some regularity, but you really have no reason to believe that at the moment. The point of the first four problems on this exam is to make some connection between functions in $W^1(a,b)$ and classically differentiable functions in $C^1(a,b)$, or at least functions with some regularity. The topic and notation are common throughout the first three problems, so I may not repeat all the assumptions of previous problems, but they should still hold.

These problems are probably a bit of a stretch for you with your current skills in mathematical analysis. I welcome you to skip down to Problem 5 if you find them too frustrating or uninteresting. The reward for tackling them is that your skills in mathematical analysis will be better when you are done. (It can also be said that many people find the content quite interesting.)

Problem 1 *As you may have conjectured in Problem 3 of Assignment 7, show that if* $u \in W^1(a,b)$ *is a weak solution of* $u' = f \in L^1_{loc}(a,b)$*, then* $y = \mu * u \in C^1(a+\delta, b-\delta)$ *is a classical solution of*

$$y' = \mu * f$$

where μ *is the standard mollifier with support* $[-\delta, \delta]$.

If we knew u was continuous, then we could recover u as the limit (as $\delta \searrow 0$) of $\mu * u$. Our strategy in Problem 2 is to determine such a limit **without knowing** u **is continuous**. So we want a limit function, but we do not have a limit. In such a situation, one uses what is called a **Cauchy estimate**.

Problem 2 *Show that given any* $\epsilon > 0$ *there is some integer* k *such that*

$$|\mu * u(x) - \tilde{\mu} * u(x)| \le \epsilon \qquad \textit{whenever } |\delta - \tilde{\delta}| < \frac{1}{k} \text{ and } a + \epsilon < x < b - \epsilon$$

where $\mu(x) = \phi(x/\delta)/\delta$ *and* $\tilde{\mu}(x) = \phi(x/\tilde{\delta})/\tilde{\delta}$ *with* ϕ *the standard bump function*

$$\phi(x) = \begin{cases} e^{1/(x^2-1)}, & |x| < 1 \\ 0, & |x| \ge 1. \end{cases}$$

Hint: Combine the integrals and apply **Hölder's inequality** *which says*

$$\int |fg| \le \max |f| \int_{\text{supp}(f)} |g|$$

when $f \in C^\infty_c(a,b)$ *and* $g \in L^1_{loc}(a,b)$.

A sequence of real numbers $a_1, a_2, a_3, \ldots$, which we will also denote by

$$\{a_j\}_{j=1}^{\infty},$$

is **Cauchy** if the following condition holds:

> Given any $\epsilon > 0$, there is some N such that whenever j and k satisfy $j, k > N$ one has
> $$|a_j - a_k| < \epsilon.$$

Notice that this condition does not require mentioning a limit of the sequence. The condition says, roughly, that the terms "bunch up" out at the end of the sequence. Notice also the (at least vague) similarity between the Cauchy condition for a sequence and the Cauchy estimate from Problem 2.

Problem 3 (a) *Show that every Cauchy sequence $\{a_j\}_{j=1}^{\infty}$ of real numbers has a well-defined limit $L \in \mathbb{R}$:*

$$\lim_{j\to\infty} a_j = L.$$

This means that for every $\epsilon > 0$, there is some N such that

$$|a_j - L| < \epsilon \qquad \text{whenever } j > N.$$

This result is called the **completeness of the real number line**.

(b) *Show that for each $x \in (a, b)$, the sequence of real numbers*

$$\left\{ j \int_{\xi\in(a,b)} \phi(j(x-\xi))\, u(\xi) \right\}_{j=1}^{\infty}$$

is Cauchy. (The terms in this sequence may not be well-defined when j is small, but since the Cauchy condition only involves terms of "high enough" index, this is not really a problem.)

(c) *Define a function $\overline{u} : (a, b) \to \mathbb{R}$ using the previous two parts of this problem. The function you define should have the property that $\overline{u}(x) = u(x)$ if u is continuous at x. Hints: This should follow from Part (d) of Problem 2 of Assignment 7. The definition of continuity at x is the following: For any $\epsilon > 0$, there is some $\delta > 0$ such that*

$$|u(\xi) - u(x)| < \epsilon \qquad \text{whenever} \qquad |\xi - x| < \delta.$$

(The number δ appearing in this definition has nothing to do with the mollification parameter.)

As the result of problems 1 through 3 above, we have obtained a function $\overline{u}$ from the weak solution u of $y' = f$. The point of the next problem is to show two things:

1. $\overline{u}$ is continuous.
2. $\overline{u}(x) = u(x)$ at "most points."

The weak solution u may not be continuous (at any point), but there exists a continuous function with which u agrees at "almost every point."

Problem 4 *Let $\overline{u}$ be the function you defined in part (c) of Problem 3 (and let μ be the standard mollifier).*

(a) *Let K be a set which is closed and compactly contained in (a, b). For example, you can take some $t > 0$ and consider $K = [a + t, b - t]$. Show that $\mu * u$ converges* **uniformly** *on K to $\overline{u}$. This means that for each $\epsilon > 0$, there is some k such that*

$$|\mu * u(x) - \overline{u}(x)| < \epsilon \qquad \text{for all } x \in K \text{ when } \delta < 1/k.$$

(b) *Show that the uniform limit of continuous functions is continuous. Conclude from this that $\overline{u} \in C^0(a, b)$.*

(c) *Again, letting K be a closed set which is compactly contained in (a, b), show that*

$$\int_K |u - \overline{u}| = 0. \tag{2}$$

Think about what this means. Two functions $f, g \in L^1_{loc}(a, b)$ satisfying $\int_K |f - g| = 0$ for $K \subset\subset (a, b)$ are said to be **equal almost everywhere** *or* **equal at almost every point** *in (a, b). Two such functions are* **essentially the same**.

Here is a (nice) linear algebra problem related to Problem 1 of Exam 2 for which I posted a solution I heartily recommend you read.

Problem 5 *Remember the vector space $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$ of linear functions $L : \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$ of $\mathbb{C}$ considered as a real vector space. The functions $L : \mathbb{C} \to \mathbb{C}$ which are linear on the vector space $\mathbb{C}$ considered as vector space (which is a field) over itself may be considered as functions in $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$. We called this subset $\mathcal{L}(\mathbb{C})$; let us continue to do so.*

(a) *It was observed in Problem 1 of Assignment 4 (at least in my solution) that $\mathcal{L}(\mathbb{C})$ consists of geometrically simple transformations of the plane $\mathbb{C}$: compositions rotations and dilations. It is not so obvious, however, how the sum of two rotations is again in this space. Consider $L_j : \mathbb{C} \to \mathbb{C}$ by*

$$L_j(z) = e^{i\theta_j} z \qquad \text{for } j = 1, 2$$

and $0 \leq \theta_1 < \theta_2 < \pi$ fixed. Find the scaling constant and rotation angle associated with the sum $L_1 + L_2$.

(b) *Find the dimension of $\mathcal{L}(\mathbb{C})$ considered as a subspace of $\mathcal{L}(\mathbb{C}_{\mathbb{R}})$.*

Here are a couple problems about solving linear systems in canonical form.

Problem 6 *Draw the phase plane diagram associated with $\mathbf{x}' = A\mathbf{x}$ when A has the canonical form*

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

in the following cases:

(a) $\lambda_1 < \lambda_2 < 0$.

(b) $\lambda_1 < 0 < \lambda_2$.

(c) $0 < \lambda_1 < \lambda_2$.

(You should solve the system too—using the decoupling—and see how your phase plane diagram relates to the initial point and form of your solution.)

Problem 7 *Solve the system* $\mathbf{x}' = A\mathbf{x}$ *when* A *has the Jordan canonical form*

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

by noting that the system partially decouples. Draw the phase plane diagram in the following cases:

(a) $\lambda < 0$.

(b) $\lambda = 0$.

(c) $\lambda > 0$.

Finally, I leave you with an opportunity to apply what you know about nonlinear systems and to learn a little more.

Problem 8 *Complete the following steps to draw the* **first quadrant** *of the phase plane diagram for the system*

$$\left\{ \begin{array}{rcl} r' & = & 0.5r(1 - 0.2r - f) \\ f' & = & 0.1f(-1 + 2r - 0.3f). \end{array} \right. \tag{3}$$

(a) *Find the equilibrium points and linearize about each one.*

(b) *Plot the nullclines and indicate the "compass direction" of the field.*

(c) *Put together what you know about the linerizations to draw representative orbits giving an accurate and detailed indication of the orbit structure and evolution of the model populations.*

(d) *Summarize in words the prediction of the model.*

A **conserved quantity** for the autonomous ODE $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is real valued function $\Phi : \mathbb{R}^n \to \mathbb{R}$ whose domain is phase space and satisfies the following condition:

Given any solution $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$,

$$\frac{d}{dt}\Phi \circ \mathbf{x}(t) \equiv 0.$$

Problem 9 *Show that if $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a conserved quantity for $\mathbf{x}' = \mathbf{F}(\mathbf{x})$, then every orbit $\mathcal{O}(\mathbf{x}_0)$ lies in the* **level set**

$$\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^n : \Phi(\mathbf{x}) = \Phi(\mathbf{x}_0)\}.$$

Note: We are using the symbol $\mathbf{x}$ in two different ways here: (1) as the dependent vector function in the ODE $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ and (2) as the independent vector variable in phase space.

Problem 10 *Consider a pendulum consisting of an arm with length ℓ having a pivot at one end and a mass m located at the other end.*

(a) *Apply Newton's second law to determine the equation of motion for a simple nonlinear frictionless pendulum having the form*

$$\theta'' = -\alpha \sin\theta.$$

Hints: Take θ to be the displacement angle of the pendulum. Assume a constant downward gravitational field and decompose the force diagram into components parallel and perpendicular to the pendulum arm. The position of the end of the pendulum may be something like:

$$\mathbf{r}(t) = \ell(\sin\theta(t), -\cos\theta(t)).$$

"Simple" here means the arm of the pendulum is assumed to have no mass and to be perfectly rigid. "Simple" also means there is a mass m concentrated at the end of the pendulum arm opposite the pivot point. Your constant α should depend on the gravitational constant and the length of the pendulum arm. (What happened to the mass?)

(b) *Write down the first order system equivalent to $\theta'' = -\alpha \sin\theta$ and find the equilibrium points. Draw pictures of the actual physical system in a state corresponding to each equilibrium.*

(c) *Show the* **total energy** *(kinetic plus potential) is a conserved quantity.*

(d) *Draw the phase plane diagram for the simple pendulum. (Don't forget that you know how to linearize at equilibrium points.)*

Problem 11 *Draw the phase plane diagram for the autonomous system*

$$\begin{cases} x' = x(1-x) \\ y' = (2x-1)y. \end{cases}$$

Hint: There is a conserved quantity.

Solution: By the chain rule

$$\frac{d}{dt}\Phi(x,y) = \frac{\partial\Phi}{\partial x}\,x' + \frac{\partial\Phi}{\partial y}\,y'.$$

On the one hand, one simple way for this to vanish is if

$$\frac{\partial\Phi}{\partial x} = y' = (2x-1)y \qquad \text{and} \qquad \frac{\partial\Phi}{\partial y} = -x' = x(x-1).$$

The first condition suggests

$$\Phi(x,y) = (x^2-x)y + c(y) = x(x-1)y + c(y)$$

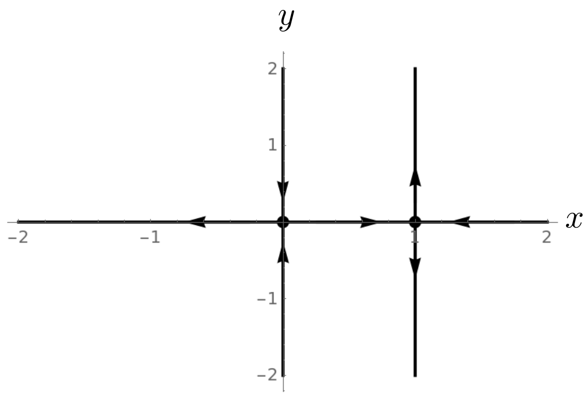
where $c = c(y)$ is an arbitrary function of y. Differentiating with respect to y, we find

$$\frac{\partial\Phi}{\partial y} = x(x-1) + c'(y)$$

which is precisely what we want when $c' = 0$. In fact, we can just take $c \equiv 0$ and observe that

$$\Phi(x,y) = (x^2-x)y \qquad \text{is conserved.}$$

This means the orbits are along the level sets of Φ. The zero level set consists of two vertical lines $x = 0$ and $x = 1$ and one horizontal line $y = 0$. Each of these lines is easily seen to consist of orbits as indicated in the figure below.



Aside from the equilibrium points at $(0, 0)$ and $(1, 0)$, the orbits are represented by solutions

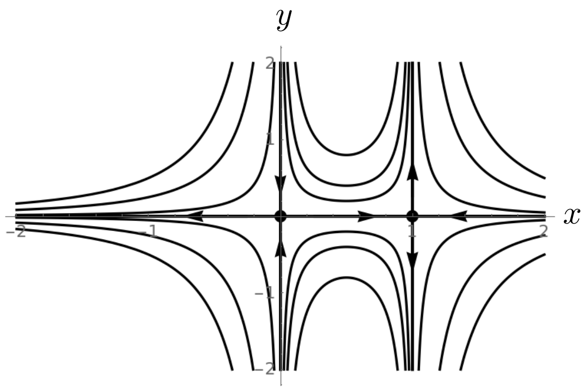
$$\begin{aligned} x(t) &\equiv 0, \quad y(t) = e^{-t} \\ x(t) &\equiv 0, \quad y(t) = -e^{-t} \\ x(t) &\equiv 1, \quad y(t) = e^{t} \\ x(t) &\equiv 0, \quad y(t) = -e^{t} \\ x(t) &= e^{2t}/(e^{2t} - 2), \quad y(t) \equiv 0 \\ x(t) &= e^{2t}/(e^{2t} + 1), \quad y(t) \equiv 0 \\ x(t) &= 2e^{2t}/1 - 2e^{2t}, \quad y(t) \equiv 0. \end{aligned}$$

Note that the existence and uniqueness theorem gives local existence and uniqueness for solutions of this system (so orbits do not cross), but the nonlinearity can be expected to (and does) lead to finite time blow-up. For example, two of the three last three solutions with orbits on the x-axis blow up in finite time. As an exercise, you could find the general solution of the first ODE (since it is decoupled), substitute it into the second equation, which will then be a linear first order equation for y, and then find the general solution of the system. As it is, we are not asked for this solution, so I will just plot the phase diagram.

We know the other orbits lie on curves $\Phi(x, y) = c$ for some nonzero constant c. This relation can be rewritten as

$$y = \frac{c}{x(x-1)}.$$

Thus, the orbits are easy to plot as indicated in the next figure below.



I didn't include the arrows on the orbits because one can easily see the field directions indicated by the orbits on the straight lines. This is also a nice system to illustrate how the linearization at the equilibrium relates to the local behavior of solutions. Notice that the global behavior is quite different. (Exercise: Do the linearizations and compare the explicit solutions of the linearized system(s) with those of the nonlinear system in this problem.)

Finally, I note that an alternative (and essentially equivalent) approach to the problem is to consider the perpendicular field

$$\mathbf{F}^{\perp} = (y(1-2x), x(1-x))$$

and the associated (possibly) exact equation

$$(x', y') \cdot \mathbf{F}^{\perp} = y(1-2x)\, x' + x(1-x)\, y' = 0.$$

which is also often written

$$y(1-2x)\, dx + x(1-x)\, dy = 0.$$

These equations have the form $P\, x' + Q\, y' = 0$ for some vector field (P, Q), and are said to be exact if there is a function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ for which the gradient satisfies

$$\nabla\Psi = D\Psi = (P, Q).$$

The necessary condition for the existence of the potential Ψ is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

This condition is easily seen to hold in this case, and this approach leads to the same (implicit) relation $\Psi(x, y) = c$ for the orbits.

Exact equations are discussed in Section 4 of Chapter 8 in Boas and also in Section 8 of Chapter 6, neither of which were mentioned in my lectures but might be worth keeping in mind for future reference.