**Math 6701, Exam 1:** (practice) Name/Section: ______________

1. (20 points) (4.8.16) Find the best linear approximation $\ell(x) = mx + b$ for the data $f(-1) = -2$, $f(0) = 0$, and $f(1) = 3$. (Hint: Use the least squares approximation method.)

**Solution:** A perfect fit would satisfy $-m + b = -2$, $b = 0$, and $m + b = 3$. That is,

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}.$$

This is not possible as one can see from row reduction of the coefficient matrix which gives

$$\begin{pmatrix} -1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

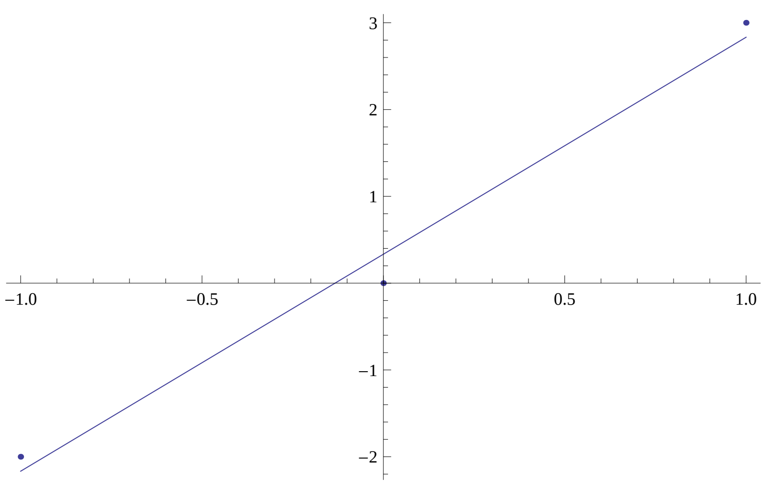
Thus, we seek the solution corresponding to projection of $(-2, 0, 3)^T$ onto the image. That is, we solve instead

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Thus, the best fit has $m = 5/2$ and $b = 1/3$.



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2. (20 points) (8.1.6) We model an evaporating substance with the assumption that the rate of evaporation is proportional to the exposed surface area. If a spherical volume evaporates so that its radius halves in six months, how long will it take for the volume to half?

**Solution:** We begin with the relation

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -4\alpha\pi r^2.$$

This tells us $r' = -\alpha$ is constant. Consequently, $r = -\alpha t + r_0$, and $-\alpha/2 + r_0 = r_0/2$ (measuring time in years). Thus, $\alpha = r_0$ and $r = r_0(1-t)$. The volume as a function of $t$ is

$$\frac{4}{3}r_0^3(1-t)^3.$$

We want to know when this quantity is $2r_0^3/3$. That is, when $(1-t)^3 = 1/2$ or $t = 1 - 1/\sqrt[3]{2} \doteq 0.2$ years, or about 2.5 months. Obviously, it will be less than 6 months. Why?

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3. (20 points) (8.6.34) Solve the initial value problem

$$\begin{cases} y'' - 5y' + 6y = 2e^x + 6x - 5 \\ y(0) = 0 = y'(0). \end{cases}$$

**Solution:** The general solution of the associated homogeneous ODE is

$$y_h(x) = ae^{2x} + be^{3x}.$$

There is no interference between this solution and the forcing terms, so we can find a particular solution of $u'' - 5u' + 6u = 2e^x$ which has the form $u = \alpha e^x$. That solution is $u = e^x$. Also, setting $v = \alpha x + \beta$, we can solve $v'' - 5v' + 6v = -5\alpha + 6\alpha x + 6\beta = 6x - 5$ with $v = x$. Thus, a particular solution is $y_p = u + v = e^x + x$. The general solution of the ODE is therefore

$$y = ae^{2x} + be^{3x} + e^x + x$$

where $a$ and $b$ are arbitrary constants. In order to get the initial conditions, we need $a + b + 1 = 0$ and $2a + 3b + 2 = 0$. That is, $b = 0$ and $a = -1$. Therefore, the solution of the initial value problem is

$$y = -e^{2x} + e^x + x.$$

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4. (20 points) (8.6.34) Use the Laplace transform to solve the initial value problem

$$\begin{cases} y'' - 5y' + 6y = 2e^x + 6x - 5 \\ y(0) = 0 = y'(0). \end{cases}$$

**Solution:** Since the initial conditions are all zero (homogeneous), the Laplace transform of the initial value problem is

$$s^2Y - 5sY + 6Y = \frac{2}{s-1} + \frac{6}{s^2} - \frac{5}{s}.$$

That is,

$$Y = \frac{2}{(s-1)(s-2)(s-3)} + \frac{6}{s^2(s-2)(s-3)} - \frac{5}{s(s-2)(s-3)}$$
$$\frac{2\mathcal{L}[e^x]}{(s-2)(s-3)} + \frac{6\mathcal{L}[x]}{(s-2)(s-3)} - \frac{5\mathcal{L}[1]}{(s-2)(s-3)}.$$

By partial fractions,

$$\frac{1}{(s-2)(s-3)} = \frac{1}{s-3} - \frac{1}{s-2} = \mathcal{L}[e^{3x}] - \mathcal{L}[e^{2x}].$$

Thus, we can use the "convolution" property of Laplace transform to get

$$\begin{aligned} Y = {} & 2\mathcal{L}\left[\int_0^x e^{x-\tau}e^{3\tau}\,d\tau\right] - 2\mathcal{L}\left[\int_0^x e^{x-\tau}e^{2\tau}\,d\tau\right] \\ & + 6\mathcal{L}\left[\int_0^x (x-\tau)e^{3\tau}\,d\tau\right] - 6\mathcal{L}\left[\int_0^x (x-\tau)e^{2\tau}\,d\tau\right] \\ & - 5\mathcal{L}\left[\int_0^x e^{3\tau}\,d\tau\right] + 5\mathcal{L}\left[\int_0^x e^{2\tau}\,d\tau\right]. \end{aligned}$$

Evaluating the integrals, we get

$$\begin{aligned} Y = {} & 2\mathcal{L}\left[e^x(e^{2x}-1)/2\right] - 2\mathcal{L}\left[e^x(e^x-1)\right] \\ & + 6\mathcal{L}\left[x(e^{3x}-1)/3\right] - 6\mathcal{L}\left[\int_0^x \tau e^{3\tau}\,d\tau\right] - 6\mathcal{L}\left[x(e^{2x}-1)/2\right] + 6\mathcal{L}\left[\int_0^x \tau e^{2\tau}\,d\tau\right] \\ & - 5\mathcal{L}\left[(e^{3x}-1)/3\right] + 5\mathcal{L}\left[(e^{2x}-1)/2\right] \\ = {} & \mathcal{L}\left[2(e^{3x}-e^x)/2 - 2(e^{2x}-e^x)\right] \\ & + 6\mathcal{L}\left[x(e^{3x}-1)/3\right] - 6\mathcal{L}\left[xe^{3x}/3 - (e^{3x}-1)/9\right] \\ & - 6\mathcal{L}\left[x(e^{2x}-1)/2\right] + 6\mathcal{L}\left[xe^{2x}/2 - (e^{2x}-1)/4\right] \\ & - 5\mathcal{L}\left[(e^{3x}-1)/3\right] + 5\mathcal{L}\left[(e^{2x}-1)/2\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} y = {} & e^{3x} + e^x - 2e^{2x} + 2xe^{3x} - 2x - 2xe^{3x} \\ & + 2e^{3x}/3 - 2/3 - 3xe^{2x} + 3x + 3xe^{2x} - 3e^{2x}/2 + 3/2 - 5e^{3x}/3 + 5/3 + 5e^{2x}/2 - 5/2 \\ = {} & e^x - e^{2x} + x. \end{aligned}$$

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5. (20 points) (8.11.9) A damped oscillator is modeled by the operator $L[y] = y'' + 2y' + 10y$, and is started in motion with the initial conditions $y(0) = 1$, $y'(0) = 0$. At some positive time $t_0$ an impulsive force stops the system so that $y(t) = 0$ for $t > t_0$. At what time can such an impulse be applied? Give also the direction and magnitude of the impulse. (Hint: Model the initial value problem with $L[y] = a\delta(t - t_0)$.)

**Solution:** The Laplace transform of the problem described above is

$$(s^2 + 2s + 10)Y - s - 2 = ae^{-st_0}.$$

Since $s^2 + 2s + 10 = (s + 1)^2 + 9$,

$$Y = \frac{s}{(s+1)^2 + 9} + \frac{2}{(s+1)^2 + 9} + \frac{ae^{-st_0}}{(s+1)^2 + 9}.$$

Shifting in $s$, we see

$$\mathcal{L}[e^{-t}\cos(3t)] = \frac{s+1}{(s+1)^2 + 9}, \quad \mathcal{L}[e^{-t}\sin(3t)] = \frac{3}{(s+1)^2 + 9}.$$

Furthermore shifting in $s$ and $t$, we find

$$\mathcal{L}[e^{-(t-t_0)}\sin(3(t - t_0))H(t - t_0)] = \frac{3e^{-st_0}}{(s+1)^2 + 9}$$

where $H$ is the Heaviside function. It follows that

$$Y = \mathcal{L}[e^{-t}\cos(3t)] + \frac{1}{3}\mathcal{L}[e^{-t}\sin(3t)] + \frac{ae^{t_0}}{3}\mathcal{L}[e^{-t}\sin(3(t - t_0))H(t - t_0)].$$

Therefore,

$$\begin{aligned} y &= \frac{e^{-t}}{3}[3\cos(3t) + \sin(3t) + ae^{t_0}\sin(3(t - t_0))H(t - t_0)] \\ &= \frac{e^{-t}}{3}[\sqrt{10}\cos(3t - \psi) + ae^{t_0}\sin(3(t - t_0))H(t - t_0)] \end{aligned}$$

where $\cos\psi = 3/\sqrt{10}$ and $\sin\psi = 1/\sqrt{10}$, that is, $\psi = \sin^{-1}(1/\sqrt{10})$. Evidently, the impulse must come at a time when the undisturbed motion passes through equilibrium. This means when $3t - \psi = \pi/2 + \pi k$ for some $k = 0, 1, 2, \ldots$. Thus, setting $t_0 = (\psi + \pi/2 + \pi k)/3$, we desire for $t > t_0$ to have

$$\sqrt{10}\cos(3t - \psi) + ae^{t_0}\sin(3t - \psi - \pi/2 + \pi k) = 0.$$

Since $\sin(3t - \psi - \pi/2 + \pi k) = -\cos(3t - \psi + \pi k) = (-1)^{k+1}\cos(3t - \psi)$, we want

$$\sqrt{10} + (-1)^{k+1}ae^{t_0} = 0,$$

or the magnitude can be

$$a = (-1)^k \sqrt{10} e^{-t_0}$$

at time $t_0 = (\psi + \pi/2 + \pi k)/3$, for some $k = 0, 1, 2, \ldots$.

The first such time would be $t_0 = (\sin^{-1}(1/\sqrt{10}) + \pi/2)/3$, and we would need a positive impulse $a = \sqrt{10} e^{-t_0}$ to stop the system. This makes sense since the system is released with positive displacement and will be moving down on the first pass through equilibrium.