Math 6701, Final Exam (practice) Name and section: ______________

1. (20 points) (3.11.19) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying $L(\mathbf{e}_1) = (1,2,2)$, $L(\mathbf{e}_2) = (2,3,0)$, and $L(\mathbf{e}_3) = (2,0,3)$ where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2\}$ is the standard basis for $\mathbb{R}^3$. Diagonalize the linear transformation L.

Solution: I will first switch the notation of $\mathbb{R}^3$ to columns, so the matrix associated to the linear transformation becomes

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix}.$$

Next, I will compute $\det(A - \lambda I)$ to find the eigenvalues. Expanding along the last column of $A - \lambda I$, we find

$$2(-2)(3-\lambda)+(3-\lambda)[(1-\lambda)(3-\lambda)-4] = (3-\lambda)[-8+\lambda^2-4\lambda+3] = (3-\lambda)(\lambda-5)(\lambda+1).$$

For $\lambda = -1$, reducing $A - \lambda I$ we find

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $v_3(-2,1,1)^T$ is a solution of the associated homogeneous system for any v_3, and $(-2,1,1)^T$ is an eigenvector.

For $\lambda = 3$, reducing $A - \lambda I$ we find

$$\begin{pmatrix} -2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $v_3(0,-1,1)^T$ is a solution of the associated homogeneous system for any v_3, and $(0,1,-1)^T$ is an eigenvector.

For $\lambda = 5$, reducing $A - \lambda I$ we find

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, $v_3(1,1,1)^T$ is a solution of the associated homogeneous system for any v_3, and $(1,1,1)^T$ is an eigenvector.

Since we have a basis of real eigenvectors, we can diagonalize. Setting

$$Q^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix},$$

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we find

$$Q = \frac{1}{6}\begin{pmatrix} -2 & 1 & 1 \\ 0 & 3 & -3 \\ 2 & 2 & 2 \end{pmatrix},$$

and

$$QAQ^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

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2. (20 points) Solve the system of ordinary differential equations

$$\begin{cases} x' = x + 2y + 2z \\ y' = 2x + 3y \\ z' = 2x + 3z. \end{cases}$$

Solution: Notice that this system can be written as

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix} \mathbf{x}.$$

In view of the previous problem, the general solution of this system is therefore

$$ae^{-t}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + be^{3t}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + ce^{5t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

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3. (20 points) (12.11.7) Use the method of Frobenius to find two linearly independent solutions of the ODE

$$x^2y'' - (x^2+2)y = 0.$$

Solution: Substituting

$$y = \sum_{j=0}^{\infty} a_j x^{j+\beta}$$

into the equation we find

$$\sum_{j=0}^{\infty}[(j+\beta-1)(j+\beta)-2]a_j x^{j+\beta} - \sum_{j=0}^{\infty} a_j x^{j+\beta+2} = 0. \tag{1}$$

The $j = 0$ (or power β) term gives the indicial equation

$$\beta^2 - \beta - 2 = (\beta+1)(\beta-2) = 0$$

corresponding to $a_0 \neq 0$. Notice we have the middle case of Frobenius' theorem in which there are two distinct roots, but they differ by an integer—and not only that, both roots are integers.

Taking $\beta = 2$ (the larger root as per the theorem) the next coefficient relation is

$$(\beta^2+\beta-2)a_1 = 4a_1 = 0$$

and we conclude $a_1 = 0$.

For $j \geq 2$, we have

$$[(j+1)(j+2)-2]a_j - a_{j-2} = (j^2+3j)a_j - a_{j-2} = 0$$

or

$$a_j = \frac{1}{j(j+3)}a_{j-2}.$$

It follows that $a_j = 0$ for all j odd, and a first solution is given by a regular power series

$$\begin{aligned}
y_1(x) &= x^2\left(1 + \frac{1}{(5)(2)}x^2 + \frac{1}{(5)(2)(7)(4)}x^4 + \frac{1}{(5)(2)(7)(4)(9)(6)}x^6 + \cdots\right) \\
&= x^2\left(1 + \sum_{k=1}^{\infty}\frac{1}{(2k+3)(2k+1)\cdots(5)(2k)(2k-2)\cdots(2)}x^{2k}\right) \\
&= x^2\sum_{k=0}^{\infty}\frac{6(k+1)}{(2k+3)!}x^{2k} \\
&= \sum_{k=0}^{\infty}\frac{6(k+1)}{(2k+3)!}x^{2k+2}.
\end{aligned}$$

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According to the Frobenius theorem, a second solution should have the form

$$y_2(x) = A\ln(x)y_1(x) + \frac{1}{x}\sum_{j=0}^{\infty} b_j x^j.$$

If we set $\phi(x) = A\ln(x)y_1(x)$ and $\psi(x) = \sum b_j x^{j-1}$, while $L[y] = x^2y'' - (x^2+2)y$, then we have

$$L[y_2] = L[\phi] + L[\psi].$$

Since $\phi' = A[y_1/x + \ln(x)y_1']$ and $\phi'' = A[-y_1/x^2 + 2y_1'/x + \ln(x)y'']$,

$$\begin{aligned} L[\phi] &= A(-y_1 + 2xy_1') + A\ln(x)L[y_1] \\ &= A(2xy_1' - y_1). \end{aligned}$$

Thus, writing $y_1 = \sum a_j x^{j+2}$ where the coefficients a_j are known, we get

$$\begin{aligned} L[\phi] &= A\sum \left[2(j+2) - 1\right] a_j x^{j+2} \\ &= A\sum_{j=0}^{\infty}(2j+3)a_j x^{j+2}. \end{aligned}$$

With the usual substitution of ψ (just look at (1) with coefficients b_j and $\beta = -1$) we find

$$\begin{aligned} L[\psi] &= \sum_{j=0}^{\infty}[(j-2)(j-1) - 2]b_j x^{j-1} - \sum_{j=0}^{\infty} b_j x^{j+1} \\ &= \sum_{j=0}^{\infty} j(j-3)b_j x^{j-1} - \sum_{j=2}^{\infty} b_{j-2}x^{j-1}. \end{aligned}$$

Adding the series expressions for $L[\phi]$ and $L[\psi]$, we see the x^{-1} term is just the indicial equation again (this time with $\beta = -1$), so we can take $b_0 \neq 0$.

The next coefficient (for the constant term) is $-2b_1 = 0$. Thus,

$$b_1 = 0.$$

The linear term coefficient is $-2b_2 - b_0$, so

$$b_2 = -\frac{1}{2}b_0,$$

and for $j \geq 3$,

$$j(j-3)b_j - b_{j-2} + A\left[2(j-3)+3\right]a_{j-3} = 0.$$

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For $j = 3$ this gives $-b_1 + 3Aa_0 = 0$. Since $b_1 = 0$ and $a_0 \neq 0$, we conclude $A = 0$, and we are reduced to a simple Froebenius series for y_2 with $\beta = -1$:

$$L[\psi] = \sum_{j=4}^{\infty} \left[j(j-3)b_j - b_{j-2} \right] x^{j-1} = 0.$$

For $j \geq 4$ therefore, we have

$$b_j = \frac{1}{j(j-3)} b_{j-2}.$$

All b_j with odd indices vanish, and a second solution is given by

$$\begin{aligned} y_2(x) &= \frac{1}{x}\left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{(6)(3)(8)}x^6 - \frac{1}{(8)(5)(6)(3)(8)}x^8 - \cdots\right) \\ &= \frac{1}{x} - \frac{x}{2} - \frac{1}{8}\sum_{k=2}^{\infty} \frac{1}{(2k)(2k-3)\cdots(6)(3)} x^{2k-1} \\ &= -\sum_{k=0}^{\infty} \frac{2k-1}{(2k)!} x^{2k-1}. \end{aligned}$$

Note: It is not often that the series turns out to have a closed form.

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4. (20 points) Find a conformal mapping ϕ of the unit disk onto the upper half plane with $\phi(\pm i) = \pm 1$.

Solution: The unit disk $\{z = a+bi : |z| < 1\}$ in $\mathbb{C}$ corresponds (under stereographic projection) to the lower half $\{(x, y, z) : x^2+y^2+z^2 = 1, z < 0\}$ of the Riemann sphere in $\mathbb{R}^3$ with

$$\pm i \mapsto (0, \pm 1, 0).$$

The upper half plane in $\mathbb{C}$ corresponds to the $+y$-half of the Riemann sphere with

$$\pm 1 \mapsto (\pm 1, 0, 0).$$

Since the upper hemisphere can be rigidly moved one the $+y$-hemisphere with $(0, \pm 1, 0) \mapsto (\pm 1, 0, 0)$, we know this conformal mapping can be obtained as a linear fractional transformation

$$\phi(z) = \frac{az + b}{cz + d}$$

for some complex constants a, b, c, d.

There are several ways to proceed from here. For example, one can proceed by (more or less) trial and error. If you have the intuition, for example, that symmetry should prevail, then you will need either $-1 \mapsto 0$ or $1 \mapsto 0$. Thus, you can take the numerator to be $a(z \pm 1)$.

A more systematic approach is to consider the rigid motion of the Riemann sphere carefully. Such consideration makes it evident that

$$\infty \mapsto -i \qquad \text{in other words } (0, 0, 1) \mapsto (0, -1, 0).$$

Similarly we must have $\phi(0) = i$ and $\phi(-1) = \infty$ and $\phi(1) = 0$. Plugging in these last two, we know $-c + d = 0$ and $a + b = 0$. Thus,

$$\phi(z) = \frac{a}{c}\,\frac{z-1}{z+1} = \alpha\,\frac{z-1}{z+1}$$

for some constant α. Using $i \mapsto 1$, this means

$$1 = \alpha\frac{i-1}{i+1} = -\alpha\frac{(1-i)^2}{2} = \alpha i.$$

Hence, $\alpha = -i$ and

$$\phi(z) = -i\frac{z-1}{z+1} = \frac{i - iz}{z+1}.$$

(You can get α more simply from $\phi(0) = i$.)

Incidentally, there are various interesting ways to decompose the rigid motions of the hemispheres into simple rotations. For example, you can first do a 90 degree

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clockwise (negative orientation or left handed) rotation about the z-axis followed by a 90 degree counterclockwise rotation about the x-axis. The first rotation corresponds to $g(z) = -iz$. The second one is

$$h(z) = -i\frac{z+i}{z-i} = \frac{1-iz}{z-i}.$$

Then you can easily check that $\phi(z) = h(g(z))$.

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5. (a) (10 points) Use the residue theorem to determine

$$\int_\gamma \frac{1}{z^2 - 2z + 2}$$

where γ is a quarter circle loop of radius $r > \sqrt{2}$ given by

$$\gamma(t) = \begin{cases} t \quad \text{for } 0 \le t \le r, \\ re^{i(t-r)} \quad \text{for } r \le t \le r + \pi/2, \\ (2r + \pi/2 - t)i \quad \text{for } r + \pi/2 \le t \le 2r + \pi/2. \end{cases}$$

(b) (10 points) Use the calculation above to determine the value of the real integral

$$\int_0^\infty \frac{t^2 - 2}{t^4 + 4}\, dt.$$

Solution:

(a) Notice that

$$\frac{1}{z^2 - 2z + 2} = \frac{1}{(z - 1 - i)(z - 1 + i)}$$

has simple poles at $z = 1 \pm i$ with $1 + i$ inside the loop. Thus,

$$\int_\gamma \frac{1}{z^2 - 2z + 2} = 2\pi i \text{Res}_{z=1+i} \frac{1}{z^2 - 2z + 2} = 2\pi i \frac{1}{1 + i - 1 + i} = \pi.$$

(b) Now we compute explicitly:

$$\int_\gamma \frac{1}{z^2 - 2z + 2} = \int_0^r \frac{1}{t^2 - 2t + 2}\, dt + \int_0^{\pi/2} \frac{rie^{it}}{r^2 e^{2it} - 2re^{it} + 2}\, dt + \int_0^r \frac{-i}{-(r-t)^2 - 2i(r-t) + 2}\, dt.$$

Noting that each of these integrals has a finite limit as $r \to \infty$ with

$$\lim_{r \nearrow \infty} \left| \int_0^{\pi/2} \frac{rie^{it}}{r^2 e^{2it} - 2re^{it} + 2}\, dt \right| \le \lim_{r \nearrow \infty} \int_0^{\pi/2} \frac{r}{r^2 - 2r - 2}\, dt \le \lim_{r \nearrow \infty} \frac{\pi}{2} \frac{r}{r^2/2}\, dt = 0,$$

we have

$$\pi = \int_0^\infty \frac{1}{t^2 - 2t + 2} + i \int_0^\infty \frac{1}{t^2 + 2it - 2}\, dt.$$

Now,

$$\frac{1}{t^2 + 2it - 2} = \frac{t^2 - 2 - 2it}{(t^2 - 2)^2 + 4t^2} = \frac{t^2 - 2 - 2it}{t^4 + 4}.$$

Thus,

$$\pi = \int_0^\infty \left[\frac{1}{t^2 - 2t + 2} + \frac{2t}{t^4 + 4} \right] dt + i \int_0^\infty \frac{t^2 - 2}{t^4 + 4}\, dt.$$

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Equating real and imaginary parts, we find

$$\int_0^\infty \left[\frac{1}{t^2-2t+2}+\frac{2t}{t^4+4}\right] dt = \pi$$

and

$$\int_0^\infty \frac{t^2-2}{t^4+4}\, dt = 0.$$