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CALCULUS OF VARIATION PROJECT

05/03/2006
A Modified Version of the "Hanging Chain Problem"

Hanging a chain connected to a rope—What shape would it take?
Outline

Problem
Introduction

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Mathematical Model and Derivation
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Mathematical Model and Derivation

Solving the equation
Outline

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Mathematical Model and Derivation

Solving the equation

Comparison with the classical problem
Problem
Let us consider a rope connected to a chain of density $\rho_2$.
Let the lengths be $L_1$ and $L_2$ respectively so that the total length is $L_1 + L_2$.
Assume that this rope connected to the chain is hanged from the end point $(a, A)$ to the end point $(b, B)$.
Let us consider a rope connected to a chain of density $\rho_2$. Let the lengths be $L_1$ and $L_2$ respectively so that the total length is $L_1 + L_2$. Assume that this rope connected to the chain is hanged from the end point $(a, A)$ to the end point $(b, B)$. Which curve will it take?
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Observe the shape of chain and the curve
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Solve the ODE of the curve and try to figure out the variables
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Compare the plotted curve to the shape of the hanging chain by an experiment

Some interesting problems
Mathematical Model and Derivation
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The problem is

\[ L(u, t) = \rho_1 \int_a^t (u - u_0) \sqrt{1 + u'^2} \, dx + \rho_2 \int_t^b (u - u_0) \sqrt{1 + u'^2} \, dx \]

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which is to be minimized by an appropriate values of \( t \), and the function \( u(t) \).

Here \( u \) is the curve, \( t \) is the connection point and the end points are \((a, A)\) and \((b, B)\). Thus, the problem can be converted into the minimization of

\[ \rho_1 \int_a^t u_1 \sqrt{1 + u'^2} \, dx + \rho_2 \int_t^b u_2 \sqrt{1 + u'^2} \, dx \]

where \( u_1 \) is the minimizer of the rope hanged from \((a, A)\) to \((t, y)\) while \( u_2 \) is the minimizer of the chain from \((t, y)\) to \((b, B)\).
Since the rope has a very small density compared to the chain it will look like a line, i.e.,
\[ u_1(x) = c(x - a) + A. \] So the problem turns out to be minimizing
\[
\rho_1 \left[ \frac{ct^2}{2} - act + At - \frac{ca^2}{2} - a^2 c + Aa \right] \sqrt{1 + c^2} + \rho_2 \int_a^x u_2 \sqrt{1 + u_2'^2} \, dx
\]
with respect to the conditions
\[
\int_a^x \sqrt{1 + u_1'^2} \, dt = L_1 \quad \int_a^b \sqrt{1 + u_2'^2} \, dt = L_2
\]
\[
 u_1(a) = A \quad u_1(t) = y = u_2(t) \quad u_2(b) = B
\]
Since \( \rho_1 / \rho_2 = 0 \), we can neglect the term with \( \rho_1 \).
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with respect to the conditions

$$\int_a^x \sqrt{1 + u_1'^2} \, dt = L_1$$
$$u_1(a) = A \quad u_1(t) = y = u_2(t)$$
$$u_2(b) = B$$

Since $\rho_1 / \rho_2 = 0$, we can neglect the term with $\rho_1$. There is one more constraint: The derivatives $u_1'$ and $u_2'$ must be equal at $t$; i.e., $u_1'(t) = u_2'(t)$.
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**Claim 1:** If $(u(t) - A)^2 + (t - a)^2 \leq L^2_1$ then there is no minimizer. This will give us that $u_1$ must be a line.
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When we consider the shape of the curve, the fact that $u_1$ is a line is obvious. We will prove it though.

**Claim 1:** If $(u(t) - A)^2 + (t - a)^2 \leq L_1^2$ then there is no minimizer. This will give us that $u_1$ must be a line.

**Claim 2:** $u'_1(t) = u'_2(t)$ at the connection point $t$. We can show this result by using the tranversality conditions.
Now, let’s consider \((a, A) = (0, 30), (b, B) = (40, 30)\), and \(L_1 = L_2 = 30\). Thus, we have

\[
u_1(x) = cx + a
\]

(2) \(u_2(x) = a \cosh \left(\frac{x-b}{a}\right) - a \cosh \left(\frac{40-b}{a}\right) + 30;\)

(3) \(u_1(0) = 30, u_2(40) = 30;\)

(4) \(u_1(t) = u_2(t)\) at the connection point \(t;\)

(5) \(u'_1(t) = u'_2(t)\) at the connection point \(t.\)
Solving the equations
◊ Remember $u_1(x) = cx + a$ and $u_2(x) = acosh\left(\frac{x-b}{a}\right) - acosh\left(\frac{40-b}{a}\right) + 30$;
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Length of the rope is 30 then \( \int_{0}^{t} \sqrt{1 + u'^2} \, dx = 30 \) and that implies \( \sqrt{1 + c^2t} - 30 = 0 \),
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Length of the chain is 30 then $\int_t^{40} \sqrt{1 + u_2'^2} \, dx = 30$ and that implies
$$asinh\left(\frac{40-b}{a}\right) - asinh\left(\frac{t-b}{a}\right) - 30 = 0,$$
\[ \text{Remember } u_1(x) = cx + a \text{ and } u_2(x) = acosh\left(\frac{x-b}{a}\right) - acosh\left(\frac{40-b}{a}\right) + 30; \]

\[ \text{Length of the rope is 30 then } \int_0^t \sqrt{1 + u_1'^2} \, dx = 30 \text{ and that implies } \sqrt{1 + c^2 t} - 30 = 0, \]

\[ \text{Length of the chain is 30 then } \int_t^{40} \sqrt{1 + u_2'^2} \, dx = 30 \text{ and that implies } asinh\left(\frac{40-b}{a}\right) - asinh\left(\frac{t-b}{a}\right) - 30 = 0, \]

\[ u_1(t) = u_2(t) \text{ implies } acosh\left(\frac{t-b}{a}\right) - acosh\left(\frac{40-b}{a}\right) - ct = 0 \]
◊ Remember $u_1(x) = cx + a$ and $u_2(x) = acosh\left(\frac{x-b}{a}\right) - acosh\left(\frac{40-b}{a}\right) + 30$;

◊ Length of the rope is 30 then $\int_0^t \sqrt{1+u_1'^2} \, dx = 30$ and that implies $\sqrt{1+c^2t} - 30 = 0$,

◊ Length of the chain is 30 then $\int_t^{40} \sqrt{1+u_2'^2} \, dx = 30$ and that implies $asinh\left(\frac{40-b}{a}\right) - asinh\left(\frac{t-b}{a}\right) - 30 = 0$,

◊ $u_1(t) = u_2(t)$ implies $acosh\left(\frac{t-b}{a}\right) - acosh\left(\frac{40-b}{a}\right) - ct = 0$

◊ $u_1'(t) = u_2'(t)$ implies $sinh\left(\frac{t-b}{a}\right) - c = 0$, 
Remember \( u_1(x) = cx + a \) and \( u_2(x) = acosh\left(\frac{x-b}{a}\right) - acosh\left(\frac{40-b}{a}\right) + 30; \)

Length of the rope is 30 then \( \int_0^t \sqrt{1 + u_1'^2} \, dx = 30 \) and that implies \( \sqrt{1 + c^2t} - 30 = 0, \)

Length of the chain is 30 then \( \int_t^{40} \sqrt{1 + u_2'^2} \, dx = 30 \) and that implies \( asinh\left(\frac{40-b}{a}\right) - asinh\left(\frac{t-b}{a}\right) - 30 = 0, \)

\( u_1(t) = u_2(t) \) implies \( acosh\left(\frac{t-b}{a}\right) - acosh\left(\frac{40-b}{a}\right) - ct = 0 \)
\( u_1'(t) = u_2'(t) \) implies \( sinh\left(\frac{t-b}{a}\right) - c = 0, \)

To solve this system of equations we used Mathematica.
For \( \{a, 4\}, \{b, 25\}, \{t, 20\}, \{c, -0.6\} \) we have
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\[
\text{FindRoot}\left\{a \sinh\left(\frac{40 - b}{a}\right) - a \sinh\left(\frac{t - b}{a}\right) - 30 == 0,
\sqrt{1 + c^2} \times t - 30 == 0,
\right. \\
\left. a \cosh\left(\frac{t - b}{a}\right) - a \cosh\left(\frac{40 - b}{a}\right) - c \times t == 0,
\sinh\left(\frac{t - b}{a}\right) - c == 0\right\},
\{a -> 5.64307, b -> 27.5197, t -> 23.3596, c -> -0.805816\}
\]
For \{a, 4\}, \{b, 25\}, \{t, 20\}, \{c, -0.6\} we have

\[\text{FindRoot}\left\{a \sinh\left(\frac{40-b}{a}\right) - a \sinh\left(\frac{t-b}{a}\right) - 30 == 0, \right.\]
\[\left.\sqrt{1+c^2} \cdot t - 30 == 0, \right.\]
\[a \cosh\left(\frac{t-b}{a}\right) - a \cosh\left(\frac{40-b}{a}\right) - c \cdot t == 0, \]
\[\sinh\left(\frac{t-b}{a}\right) - c == 0\},\]
\[\{a \rightarrow 5.64307, b \rightarrow 27.5197, t \rightarrow 23.3596, c \rightarrow -0.805816\}\]
Here is the first graph:
Here is the first graph:
Here is the second graph:
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Conclusion

I. An interesting problem is connecting two chains with different densities successively and see what shape would it take.
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I. An interesting problem is connecting two chains with different densities successively and see what shape would it take.

II. Another interesting problem would be connecting a chain to another not from the end points as in the picture.
The End

Thank You