The shape of a hanging chain

a project in the calculus of variations

April 15, 2018

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Chapter 1

Introduction

If a chain or cable has its ends fixed at two different points and hangs under the influence of gravity, it takes the shape of a hyperbolic cosine curve. We now describe this shape precisely and explain how it arises as a minimizer of potential energy among many possible shapes.

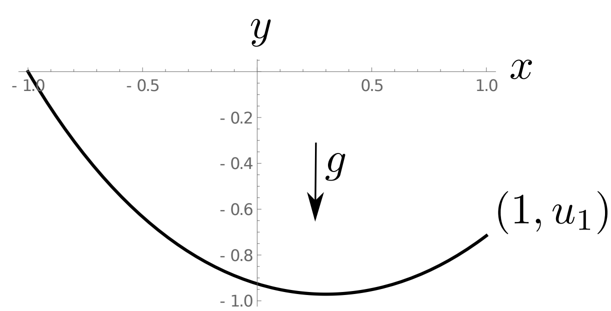

Figure 1.1: the shape of a chain hanging from its endpoints in gravity

Chapter 2

Analysis

2.1 Model

Let $\ell > 0$ be the length of the chain and let ρ denote the linear density of mass along the length of chain. Choose x, y-coordinates with the left end of the chain fixed at $(-1, 0)$ and the right end at $(1, u_1)$. We have made a choice of units here so that the horizontal distance between the fixed endpoints is 2 units. This is equivalent to scaling the system given in some particular initial units. We could also assume u_1 has a specific sign, say $u_1 > 0$, but this is not necessary.

Given the length constraint on the chain, we must have

$$1 + u_1^2 < \ell^2. \tag{2.1}$$

There are many curves of length ℓ connecting $(-1, 0)$ to $(1, u_1)$. Among these consider C^1 curves given by the graph of a function $u : [-1, 1] \to \mathbb{R}$. The length constraint may then be written as

$$\int_{-1}^{1} \sqrt{1 + [u'(x)]^2}\, dx = \ell.$$

Assuming a constant gravitational field $\vec{G} = -g(0, 1)$ and zero potential at $y = 0$, we may integrate to approximate the potential energy of a portion of the chain having mass $\Delta m_j = \rho\sqrt{1 + [u'(x_j^*)]^2}\, \Delta x_j$:

$$\text{approximate potential energy } V_j \ = \int_0^{u(x_j^*)} \rho g \sqrt{1 + u'(x_j^*)^2}\, \Delta x_j\, dy.$$

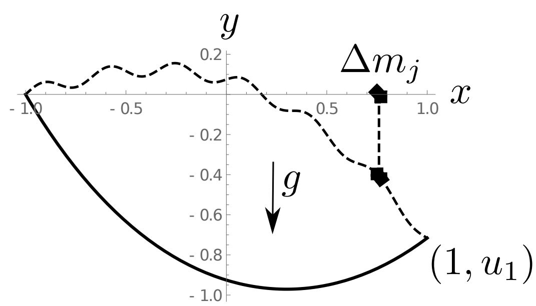

Figure 2.1: an alternative chain shape and the associated potential energy

The potential energy associated with a point mass is given by the work required to move the mass from a position of zero potential to another position, that is, $-\int_\gamma F \cdot T$ where F is the force field, γ is a path connecting a position of zero potential to the position of the mass, and T is the unit tangent vector along the path. In this case the force $F = \Delta m_j \vec{G} = -\Delta m_j g(0,1)$ is assumed constant, and the integral amounts to the force multiplied by the vertical distance to equilibrium:

$$V_j = \rho g u(x_j^*) \sqrt{1 + u'(x_j^*)^2}\, \Delta x_j.$$

Summing over all model portions of chain and taking the limit as the maximum portion length tends to 0, we find an expression for the total potential energy as a function of the chain shape determined by u:

$$\text{potential energy } V = \lim \sum \rho g u(x_j^*) \sqrt{1 + u'(x_j^*)^2}\, \Delta x_j = \int_{-1}^{1} \rho g u(x) \sqrt{1 + u'(x)^2}\, dx.$$

By the Leibniz'/Maupertuis' principle of virtual work, or Hamilton's action principle, the observable shape u should be a critical point for

$$V[u] = \int_{-1}^{1} \rho g u(x) \sqrt{1 + u'(x)^2}\, dx$$

subject to the constraint

$$L[u] = \int_{-1}^{1} \sqrt{1 + u'(x)^2}\, dx = \ell.$$

Under the assumption that ρ and g are positive constants, we may replace the expression for V above with

$$V[u] = \int_{-1}^{1} u(x)\sqrt{1 + u'(x)^2}\, dx$$

Introducing a Lagrange multiplier λ associated with the constraint and assuming the existence of the model shape within the admissible class

$$\mathcal{A} = \{u \in C^2[-1, 1] : u(-1) = 0,\, u(1) = u_1\},$$

we set $\mathcal{F} = V + \lambda L$ and obtain the necessary condition

$$\delta\mathcal{F}_u[\phi] = \frac{d}{d\epsilon}\int_{-1}^{1} (u + \epsilon\phi + \lambda)\sqrt{1 + (u' + \epsilon\phi')^2}\, dx\Big|_{\epsilon=0} = 0$$

for all $\phi \in C_c^\infty(-1, 1)$. Differentiating under the integral and evaluating, we find

$$\int_{-1}^{1} \left[\phi\sqrt{1 + u'^2} + (u + \lambda)\frac{u'\phi'}{\sqrt{1 + u'^2}}\right] dx = 0.$$

We may integrate by parts in the second term to obtain

$$\int_{-1}^{1} \left[-\left(\frac{(u + \lambda)u'}{\sqrt{1 + u'^2}}\right)' + \sqrt{1 + u'^2}\right] \phi = 0 \quad \text{for all } \phi \in C_c^\infty(-1, 1).$$

Finally, we may apply the fundamental lemma of the calculus of variations to obtain a two point boundary value problem for a second order nonlinear ordinary differential equation for the observed shape u:

$$\left(\frac{(u + \lambda)u'}{\sqrt{1 + u'^2}}\right)' = \sqrt{1 + u'^2}, \qquad u(-1) = 0,\; u(1) = u_1.$$

We know this equation is satisfied even under the assumption $u \in C^1[-1, 1]$.

2.2 Extremal graphs

Using the assumed regularity of the observed shape u, we can also write

$$(u + \lambda)\frac{u''}{(1 + u'^2)^{3/2}} + \frac{u'^2}{\sqrt{1 + u'^2}} = \sqrt{1 + u'^2}$$

or

$$(u+\lambda)u'' = 1 + u'^2.$$

Under the assumption $u''(-1) > 0$, which (based on observation of the shape of actual physical hanging chains) seems rather reasonable, we can solve for the Lagrange multiplier and find

$$\lambda = \frac{1 + u'(-1)^2}{u''(-1)} > 0.$$

More generally, whenever $u + \lambda \neq 0$, we can write

$$\frac{u'}{1+u'^2}\, u'' = \frac{1}{u+\lambda}\, u'.$$

In particular, integrating from $x = -1$ to x,

$$\int_{u'(-1)}^{u'} \frac{t}{1+t^2}\, dt = \int_{u(-1)}^{u} \frac{1}{t+\lambda}\, dt$$

or

$$\frac{1}{2}\left[\ln(1+u'^2) - \ln(1+u'(-1)^2)\right] = \ln(u+\lambda) - \ln\lambda.$$

It follows that

$$\frac{1+u'^2}{1+u'(-1)^2} = \left(\frac{u}{\lambda} + 1\right)^2. \tag{2.2}$$

Let us pause at this point to consider the first integral equation

$$u' F_p(u, u') - F(u, u') = -c \tag{2.3}$$

where c is some constant and $F(z, p) = (z + \lambda)\sqrt{1+p^2}$ is the Lagrangian associated with $\mathcal{F}$. We have used $-c$ instead of c here to simplify things later. After a computation, we find

$$\frac{u'^2}{\sqrt{1+u'^2}} - \sqrt{1+u'^2} = -\frac{c}{u+\lambda}.$$

That is,

$$\sqrt{1+u'^2} = \frac{1}{c}(u+\lambda).$$

Taking the constant $c = \lambda/\sqrt{1+u'(-1)^2}$, which it must be, we see several things. First of all, any solution of the first integral equation with $c \neq 0$

will give a solution of (2.2). It is possible to get a solution of (2.3) with the choice $c = 0$, but in this case, we must take $u \equiv -\lambda = 0$, and we must therefore have $u_1 = 0$. This is, indeed, not a solution of the Euler-Lagrange equation for $\mathcal{F} = V + \lambda L$, but this possibility represents the exceptional case of Proposition 1.17 in the book of Buttazzo, Giaquinta, and Hildebrandt [1] in which the constraint is degenerate. In this case, the solution $u \equiv 0$ gives the shortest path between $(-1, 0)$ and $(1, u_1) = (1, 0)$ and is, therefore, a critical point for the length functional L providing the constraint. When $c \neq 0$, we obtain from the first integral equation a global justification for our assumption

$$u + \lambda \neq 0.$$

This is because every solution of the Euler-Lagrange equation must be a solution of the first integral equation. Only the solution $u \equiv 0$ in the case $u_1 = 0$ and $\ell = 2$ is exceptional.

Finally, the first integral equation tells us something about the sign of $u + \lambda$ because

$$\sqrt{1 + u'^2} = \frac{\sqrt{1 + u'(-1)^2}}{\lambda}(u + \lambda).$$

It follows that $u+\lambda$ and λ must share the same sign, and under our, seemingly justified, assumption $u''(-1) > 0$, that sign is positive. Thus, we may proceed to solve either the Euler-Lagrange equation or the first integral equation under this assumption. Making the substitution $v = (u+\lambda)\sqrt{1 + u'(-1)^2}/\lambda$, we find

$$u' = \pm\sqrt{v^2 - 1} \qquad \text{or} \qquad \frac{\lambda}{\sqrt{1 + u'(-1)^2}} v' = \pm\sqrt{v^2 - 1}.$$

It follows that

$$\cosh^{-1} v - \cosh^{-1} v(-1) = \pm\frac{\sqrt{1 + u'(-1)^2}}{\lambda}(x + 1),$$

$$v = \frac{\sqrt{1 + u'(-1)^2}}{\lambda}(u+\lambda) = \cosh\left[\pm\frac{\sqrt{1 + u'(-1)^2}}{\lambda}(x + 1) + \cosh^{-1} v(-1)\right],$$

or

$$u = -\lambda + \frac{\lambda}{\sqrt{1 + u'(-1)^2}} \cosh\left[\frac{\sqrt{1 + u'(-1)^2}}{\lambda}(x + 1) \pm \cosh^{-1}\sqrt{1 + u'(-1)^2}\right].$$

This looks rather complicated, but it does tell us that the extremals have the form of hyperbolic cosine curves. This also confirms that the constant c from the first integral equation should be positive with $c < 0$ extremals corresponding to maximizers of the energy. Substituting the value of c from the first integral equation and differentiating, we also see

$$u' = \sinh\left((x+1)/c \pm \cosh^{-1}\sqrt{1+u'(-1)^2}\right).$$

This allows us to nominally locate the vertex or lowest point on the hyperbolic cosine curve which occurs for

$$x = \mu = -1 \mp c\cosh^{-1}(\lambda/c).$$

In terms of this parameter, the extremals may be written as

$$u = -\lambda + c\cosh\left(\frac{x-\mu}{c}\right).$$

There are now three unknown parameters λ, μ, and c, but the initial condition $u(-1) = 0$ implies

$$\lambda = c\cosh\left(\frac{1+\mu}{c}\right)$$

and

$$u = c\cosh\left(\frac{x-\mu}{c}\right) - c\cosh\left(\frac{1+\mu}{c}\right).$$

The other endpoint condition takes the symmetric form

$$c\cosh\left(\frac{1-\mu}{c}\right) - c\cosh\left(\frac{1+\mu}{c}\right) = u_1.$$

Another equation we can use to determine the parameters c and μ is given by the length constraint $L[u] = \ell$.

$$u' = \sinh\left(\frac{x-\mu}{c}\right) \qquad \text{and} \qquad 1 + u'^2 = \cosh^2\left(\frac{x-\mu}{c}\right).$$

Therefore,

$$L[u] = \int_{-1}^{1} \sqrt{1+u'^2}\,dx = \int_{-1}^{1} \cosh\left(\frac{x-\mu}{c}\right)\,dx,$$

and writing down $L[u] = \ell$ we are led to the fundamental symmetric system:

$$c\cosh\left(\frac{1-\mu}{c}\right) - c\cosh\left(\frac{1+\mu}{c}\right) = u_1. \qquad (2.4)$$

and

$$c\sinh\left(\frac{1-\mu}{c}\right) + c\sinh\left(\frac{1+\mu}{c}\right) = \ell. \qquad (2.5)$$

In this symmetric form, it is possible to eliminate μ as follows: Square both equations and subtract the first from the second, noting $\ell^2 - u_1^2 \geq 4$. We get

$$c^2\left[-2 + 2\cosh\left(\frac{1-\mu}{c}\right)\cosh\left(\frac{1+\mu}{c}\right) + 2\sinh\left(\frac{1-\mu}{c}\right)\sinh\left(\frac{1+\mu}{c}\right)\right] = \ell^2 - u_1^2.$$

That is,

$$-1 + \cosh\left(\frac{2}{c}\right) = 1 + \cosh^2\left(\frac{1}{c}\right) + \sinh^2\left(\frac{1}{c}\right) = \frac{\ell^2 - u_1^2}{2c^2}.$$

That is,

$$c\sinh\left(\frac{1}{c}\right) = \frac{1}{2}\sqrt{\ell^2 - u_1^2} > 1. \qquad (2.6)$$

In this way, we obtain a single transcendental equation for c. One can show $c\sinh(1/c)$ is monotone decreasing in c for $c > 0$ and takes every value greater than 1. Let us verify the equivalent assertions for the function $f(z) = \sinh z/z$. First of all if $z \searrow 0$, we have by L'Hopital's rule

$$\lim_{z\searrow 0}\frac{\sinh z}{z} = \lim_{z\searrow 0}\cosh z = 1 \quad \text{and} \quad \lim_{z\nearrow\infty}\frac{\sinh z}{z} = \lim_{z\nearrow\infty}\cosh z = \infty.$$

Also,

$$f'(z) = \frac{z\cosh z - \sinh z}{z^2}.$$

Setting $f_1(z) = z\cosh z - \sinh z$ we see $f_1(0) = 0$ and $f_1'(z) = z\sinh z > 0$ for $z > 0$. In particular, $f_1(z) > 0$ for $z > 0$, so $f'(z) > 0$ for $z > 0$. Also,

$$\lim_{z\searrow 0} f'(z) = \lim_{z\searrow 0}\frac{f_1'(z)}{2z} = 0.$$

We have shown that f takes every value on $[1, \infty)$ uniquely and has a well-defined inverse on that interval. Thus, we have a unique solution

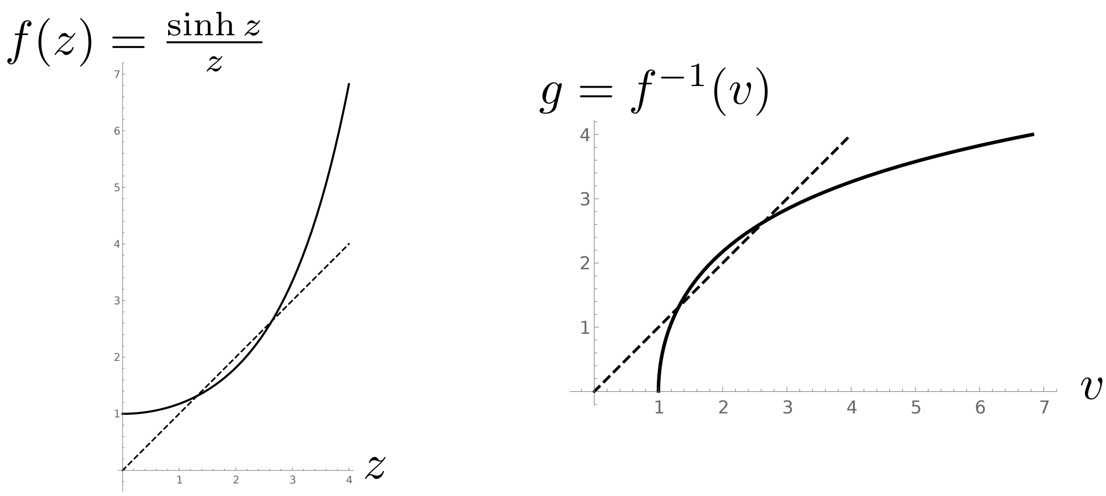

Figure 2.2: $\sinh z/z$ and its inverse

$$c = \frac{1}{f^{-1}\left(\frac{1}{2}\sqrt{\ell^2 - u_1^2}\right)}.$$

Once we know $c > 0$, we can expand (2.4) to see

$$-2c \sinh\left(\frac{1}{c}\right) \sinh\left(\frac{\mu}{c}\right) = u_1.$$

Therefore, substituting from (2.6),

$$\mu = -c \sinh^{-1}\left(\frac{u_1}{\sqrt{\ell^2 - u_1^2}}\right) \tag{2.7}$$

This completes our initial analysis of the extremals for the problem of determining the shape of a hanging chain. We summarize our discussion as follows:

Theorem 1. *For each length $\ell > 2$ and each value $u_1 \in \mathbb{R}$ with $u_1^2 < \ell^2 - 4$, there are unique values $c > 0$ and μ satisfying*

$$c \sinh\left(\frac{1}{c}\right) = \frac{1}{2}\sqrt{\ell^2 - u_1^2} \quad \text{and} \quad \mu = -c \sinh^{-1}\left(\frac{u_1}{\sqrt{\ell^2 - u_1^2}}\right)$$

such that

$$u(x) = c\cosh\left(\frac{x-\mu}{c}\right) - c\cosh\left(\frac{1+\mu}{c}\right), \quad -1 \le x \le 1$$

is the unique convex extremal for the functional

$$\mathcal{F}[u] = \int_{-1}^{1} [u(x) + \lambda]\sqrt{1 + u'(x)^2}\, dx$$

in $\mathcal{A} = \{u \in C^1[-1,1] : u(-1) = 0,\ u(1) = u_1\}$ *subject to the constraint*

$$L[u] = \int_{-1}^{1} \sqrt{1 + u'(x)^2}\, dx = \ell$$

and with

$$\lambda = c\cosh\left(\frac{1+\mu}{c}\right).$$

The uniqueness extends generally to the functional $\mathcal{F}$ *(or more precisely to* λ*) in the sense that for other values of* λ*, there is no extremal of the resulting functional for the constrained problem.*

In practice, some analysis is required to set up the computation approximating solutions of the transcendental equation for c. We include this analysis in the section on computation and now address the minimality of these extremals.

Note 1. *If* c *and* λ *are allowed to be negative negative, one finds concave extremals (satisfying* $u'' < 0$*) whose gravitational potential energy is greater than the convex solutions.*

2.3 Minimality of extremals

We have established the existence of a unique (convex) catenary extremal given by the graph of a function $u \in C^\infty[-1,1]$ and satisfying

$$u(-1) = 0, \quad u(1) = u_1, \qquad \text{and} \qquad \int_{-1}^{1} \sqrt{1 + u'^2}\, dx = \ell.$$

The function u satisfies

$$u(x) = c\cosh\left(\frac{x-\mu}{c}\right) - c\cosh\left(\frac{1+\mu}{c}\right) \tag{2.8}$$

where $c > 0$ is the unique solution of $c\sinh(1/c) = \sqrt{\ell^2 - u_1^2}\Big/ 2 > 0$, and

$$\mu = -c\sinh^{-1}\left(\frac{u_1}{\sqrt{\ell^2 - u_1{}^2}}\right).$$

We now wish to establish the following result.

Theorem 2. *The function u given in (2.8) is the unique minimizer of*

$$V[u] = \int_{-1}^{1} u\sqrt{1+u'^2}\,dx$$

on

$$\mathcal{A} = \{u \in C^1[-1,1] : u(-1) = 0,\ u(1) = u_1\}$$

subject to

$$L[u] = \int_{-1}^{1} \sqrt{1+u'^2}\,dx = \ell.$$

A fundamental difficulty in establishing this result is that the Lagrangian $F(z,p) = (z+\lambda)\sqrt{1+p^2}$ associated with the augmented functional $\mathcal{F} = V + \lambda L$ where

$$\lambda = c\cosh\left(\frac{1+\mu}{c}\right) > 0$$

is not (always) convex. Showing this is Problem 30 of Chapter 3 in the book [2] of Troutman. Following Troutman, we take the special case $u_1 = 0$. In this case $\mu = 0$, and the extremal is given by

$$u(x) = c\cosh\left(\frac{x}{c}\right) - \lambda \qquad \text{with} \qquad \lambda = c\cosh\left(\frac{1}{c}\right).$$

On the other hand, the function $u_0 \equiv 0$ satisfies $u_0 \in \mathcal{A}$, and $\delta\mathcal{F}_u[v] \equiv 0$. Taking $v = -u$, we have $u + v = u_0$ and showing $\mathcal{F}$ is **not** convex amounts to showing

$$\mathcal{F}[u_0] - \mathcal{F}[u] < 0$$

(under some circumstances). In fact,

$$\begin{aligned}
\mathcal{F}[u_0]-\mathcal{F}[u] &= \int_{-1}^{1}\lambda\,dx-\int_{-1}^{1}(u+\lambda)\sqrt{1+u'^2}\,dx \\
&= 2c\cosh\left(\frac{1}{c}\right)-c\int_{-1}^{1}\cosh^2\left(\frac{x}{c}\right)\,dx \\
&= 2c\cosh\left(\frac{1}{c}\right)-\frac{c}{2}\int_{-1}^{1}\left[\cosh\left(\frac{2x}{c}\right)+1\right]\,dx \\
&= 2c\cosh\left(\frac{1}{c}\right)-\frac{c^2}{2}\sinh\left(\frac{2}{c}\right)-c \\
& c\left[2\cosh\left(\frac{1}{c}\right)-c\sinh\left(\frac{1}{c}\right)\cosh\left(\frac{1}{c}\right)-1\right].
\end{aligned}$$

Since $x\sinh x\to\infty$ as $x\nearrow\infty$, we see that for $c>0$ small enough

$$c\sinh\left(\frac{1}{c}\right)>2,$$

and $\mathcal{F}[u_0]-\mathcal{F}[u]<0$. Recalling that c is determined by

$$c\sinh\left(\frac{1}{c}\right)=\frac{1}{2}\sqrt{\ell^2-u_1^2}=\frac{\ell}{2},$$

we find nonconvexity for chains of any length $\ell>4$.

In spite of this nonconvexity, Troutman suggests a rephrasing of the problem which leads to a much stronger result than Theorem 2 above. The function u determines a parametric curve parameterized by arclength. This is given by the function $\mathbf{x}\in C^1([0,\ell]\to\mathbb{R}^2)$ by $\mathbf{x}(s)=(\xi(s),\eta(s))$ where

$$\begin{cases}\xi(s)=\mu+c\sinh^{-1}\left[\frac{s}{c}-\sinh\left(\frac{1+\mu}{c}\right)\right]\\ \eta(s)=u(\xi(s))=c\cosh\left(\sinh^{-1}\left[\frac{s}{c}-\sinh\left(\frac{1+\mu}{c}\right)\right]\right)-c\cosh\left(\frac{1+\mu}{c}\right).\end{cases}\tag{2.9}$$

This parametric map $\mathbf{x}$ also satisfies

$$|\mathbf{x}'|\equiv 1\qquad\text{and}\qquad 2=\int_{-1}^{\ell}\xi'(s)\,ds=\int_0^{\ell}\sqrt{1-\eta'^2}\,ds.$$

Now if we let $\mathbf{x}=(\xi,\eta)\in C^1([0,\ell]\to\mathbb{R}^2)$ be any parametric curve parameterized by arclength ($|\mathbf{x}'|\equiv 1$) with $\mathbf{x}(0)=(-1,0)$ and $\mathbf{x}(\ell)=(1,u_1)$, then the potential energy expression

$$V[u]=\int_{-1}^{1}u\sqrt{1+u'^2}\,dx$$

generalizes to

$$V_1[\mathbf{x}] = \int_0^\ell \eta \, ds.$$

To see this, we may again consider a portion of chain of mass $\Delta m_j = \rho \Delta s_j$ located at a point $\mathbf{x}(s_j^*)$. The potential energy of this particular section of chain is approximately

$$\int_0^\eta \rho g \Delta s_j \, dy = \rho g \eta \Delta s_j.$$

Summing over a partition of such portions and taking the limit as the maximum length Δs_j tends to zero (and dividing out by the constant ρg as usual), we arrive at the expression for V_1 above. The following result treats these

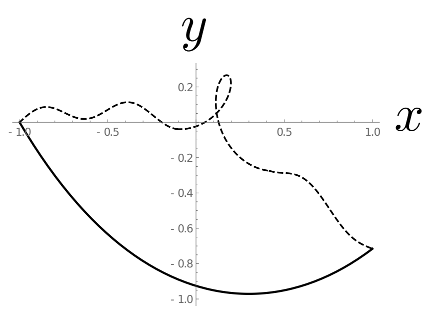

Figure 2.3: parametric chain shape: These shapes are also not required to satisfy $-1 \leq \xi \leq 1$ though the one illustrated does. (Actually, this shape has length a little longer than the original catenary chain shape.)

general parametric curves of length ℓ connecting $(-1, 0)$ to $(1, u_1)$ and asserts that the catenary graph extremal is the unique minimizer among such curves.

Theorem 3. *The catenary graph satisfying (2.9) is the unique minimizer of*

$$V_1[\mathbf{x}] = \int_0^\ell \eta \, ds$$

on

$$\mathcal{B} = \{\mathbf{x} \in C^1([0, \ell] \to \mathbb{R}^2) : \mathbf{x}(0) = (-1, 0),\ \mathbf{x}(\ell) = (1, u_1),\ |\mathbf{x}'| \equiv 1\}$$

subject to

$$L_1[\mathbf{x}] = \int_0^\ell \sqrt{1-\eta'^2}\, ds = 2.$$

Finally, we simplify the previous result slightly and prove something even more general. It will be noted that the functionals appearing above only depend on the second coordinate function of $\mathbf{x}$, namely, $\eta \in C^1[0,\ell]$. Thus, it makes sense to extend their domains and rename them:

$$V_1 : C^1[0,\ell] \to \mathbb{R} \qquad \text{by} \qquad V_1[\eta] = \int_0^\ell \eta\, ds$$

and

$$L_1 : \{\eta \in C^1[0,\ell] : |\eta'(s)| \le 1 \text{ for } 0 \le s \le \ell\} \to \mathbb{R} \qquad \text{by} \qquad L_1[\mathbf{x}] = \int_0^\ell \sqrt{1-\eta'^2}\, ds.$$

We now state the main result.

Theorem 4. *The second component of the parametric map defined in (2.9) is the unique minimizer of*

$$V_1[\eta] = \int_0^\ell \eta\, ds$$

on

$$\mathcal{B} = \{\mathbf{x} \in C^1([0,\ell] \to \mathbb{R}^2) : \eta(0) = 0,\ \eta(\ell) = u_1\}$$

subject to

$$L_1[\eta] = \int_0^\ell \sqrt{1-\eta'^2}\, ds = 2.$$

Notice the absence of the condition $|\mathbf{x}'| \equiv 1$ in the definition of $\mathcal{B}$. Notice, furthermore, that the functional L_1 is not (even) defined on all of $\mathcal{B}$, but only on

$$\mathcal{B}_1 = \{\eta \in \mathcal{B} : |\eta'(s)| \le 1 \text{ for } 0 \le s \le \ell\}.$$

Proof of Theorem 4: We show first that η from (2.9) is the unique minimizer of

$$\mathcal{G}[\eta] = (V_1 - cL_1)[\eta] = \int_0^\ell \left[\eta - c\sqrt{1-\eta'^2}\right] ds$$

on $\mathcal{B}_1$ (without constraint). This follows from two facts

1. The augmented functional $\mathcal{G} = V_1 - cL_1$ is strictly convex on $\mathcal{B}_1$ in the sense that $\mathcal{G}[\eta+v] - \mathcal{G}[\eta] \geq \delta\mathcal{G}_\eta[v]$ whenever η, $\eta+v \in \mathcal{B}_1$ with equality only if $v \equiv 0$.

2. The function η from (2.9) is an extremal for $\mathcal{G}$, that is $\delta\mathcal{G}_\eta[v] = 0$ whenever $\eta + v \in \mathcal{B}_1$.

If we can establish these two assertions, we may apply the following (easy) theorem on minimizing convex functionals:

Theorem 5. *If $\mathcal{G}$ is strictly convex on $\mathcal{B}_1$ and for every $v \in C^1[-1,1]$ such that $\eta_0 + v \in \mathcal{B}_1$ we have*

$$\delta\mathcal{G}_{\eta_0}[v] = 0$$

then we have $\mathcal{G}[\eta_0] \leq \mathcal{G}[\eta]$ for all $\eta \in \mathcal{B}_1$.

The strict convexity does not follow immediately because the augmented Lagrangian $G(z,p) = z - c\sqrt{1-p^2}$ is not strictly second order convex. We do have

$$D^2G = \begin{pmatrix} 0 & 0 \\ 0 & \frac{c}{(1-p^2)^{3/2}} \end{pmatrix}.$$

Therefore, for each $v \in C^1[0,\ell]$ such that $\eta + v \in \mathcal{B}_1$, we have

$$\begin{aligned} G(\eta+v, \eta'+v') - G(\eta,\eta') &= G_z(\eta,\eta')v + G_p(\eta,\eta')v' + \frac{c}{2(1-p_*^2)^{3/2}}v'^2 \\ &\geq G_z(\eta,\eta')v + G_p(\eta,\eta')v' \end{aligned}$$

with equality only if $v' = 0$ (pointwise). Integrating this inequality

$$\mathcal{G}[\eta+v] - \mathcal{G}[\eta] = \int_0^\ell [G_z(\eta,\eta')v + G_p(\eta,\eta')v']\,ds + \frac{c}{2}\int_0^\ell \frac{v'^2}{(1-p_*^2)^{3/2}}\,ds \geq \delta\mathcal{G}_\eta[v]$$

with equality only if $v' \equiv 0$. But if $\eta + v \in \mathcal{B}_1$, then $v(0) = v(\ell) = 0$, so equality implies $v \equiv 0$. This establishes the strict convexity of $\mathcal{G}$.

On the other hand, the Euler-Lagrange equation for $\mathcal{G}$ is

$$c\left(\frac{\eta'}{\sqrt{1+\eta'^2}}\right)' = 1$$

where the derivatives are with respect to the arclength s. To compute this for the function η from the arclength parameterization of the catenary we observe first that

$$s = \int_0^\xi \sqrt{1+u'^2}\,dx = c\sinh\left(\frac{\xi-\mu}{c}\right) + c\sinh\left(\frac{1+\mu}{c}\right).$$

Therefore,

$$\frac{d\xi}{ds} = \frac{1}{\cosh\left(\frac{\xi-\mu}{c}\right)}.$$

Having made this observation/calculation we have from (2.9)

$$\eta'(s) = \frac{du}{dx}(\xi)\frac{d\xi}{ds} = \frac{\sinh\left(\frac{\xi-\mu}{c}\right)}{\cosh\left(\frac{\xi-\mu}{c}\right)}.$$

Therefore,

$$\frac{d}{ds}\left(\frac{\eta'}{\sqrt{1-\eta'^2}}\right) = \frac{d}{dx}\left(\sinh\left(\frac{\xi-\mu}{c}\right)\right)\Big|_{x=\xi}\frac{d\xi}{ds} = \frac{1}{c},$$

and η is a C^2 classical extremal for $\mathcal{G}$. In particular, $\delta\mathcal{G}_\eta[v]\equiv 0$, and $\mathcal{G}[\eta+v]-\mathcal{G}[\eta]\geq 0$ whenever $\eta+v\in\mathcal{B}_1$ with equality only if $v\equiv 0$.

The usual argument of Theorem 5 now applies. That is, it happens that

$$L_1[\eta] = \int_0^\ell \sqrt{1-\eta'^2}\,ds = 2,$$

so for any $v\in C^1[0,\ell]$ such that $\eta+v\in\mathcal{B}$ and for which $L_1[\eta+v]=2$, we have

$$V_1[\eta+v]-cL_1[\eta+v] = \mathcal{G}[\eta+v] \geq \mathcal{G}[\eta] = V_1[\eta]-cL_1[\eta]$$

with equality only if $v\equiv 0$. Since $L_1[\eta+v]=L_1[\eta]=2$, we have

$$V_1[\eta+v]\geq V_1[\eta] \quad \text{with equality only if } v\equiv 0.$$

This establishes Theorem 4. □

Proof of Theorem 3: If $\tilde{\mathbf{x}}=(\tilde{\xi},\tilde{\eta})\in\mathcal{B}$ satisfies

$$\int_0^\ell \sqrt{1-\tilde{\eta}'^2}\,ds = 2$$

and $\mathbf{x}$ is the parametric catenary, then $\tilde{\eta} \in \mathcal{B}_1 \subset \mathcal{B}$ and satisfies $L_1[\tilde{\eta}] = 2$. Thus, by Theorem 4

$$V_1[\tilde{\mathbf{x}}] = V_1[\tilde{\eta}] \geq V_1[\eta] = V_1[\mathbf{x}] \quad \text{with equality only if } \tilde{\eta} \equiv \eta.$$

We have, in particular, $V_1[\tilde{\mathbf{x}}] \geq V_1[\mathbf{x}]$ for all $\tilde{\mathbf{x}} \in \mathcal{B}$ satisfying the constraint

$$L_1[\tilde{\mathbf{x}}] = 2.$$

In the case of equality we have $\tilde{\xi}' = \pm\sqrt{1 - \eta'^2}$ and

$$2 = \int_0^\ell \tilde{\xi}' \, ds = \int_0^\ell \sqrt{1 - \eta'^2} \, ds.$$

Since $\eta'(s) = 1$ for at most one arclength s, we conclude $\tilde{\xi} = \sqrt{1 - \eta'^2}$ and $\tilde{\mathbf{x}} \equiv \mathbf{x}$. □

Finally we prove the initial (and weakest) assertion.

Proof of Theorem 2: If $\tilde{u} \in \mathcal{A}$ and

$$L[\tilde{u}] = \int_{-1}^1 \sqrt{1 - \tilde{u}'^2} \, dx = \ell,$$

then the graph of $\tilde{u}$ may be parameterized by arclength to give a parameterized curve $\tilde{\mathbf{x}} \in \mathcal{B}$ satisfying the constraint

$$L_1[\tilde{\mathbf{x}}] = \int_0^\ell \sqrt{1 - \tilde{\eta}'^2} \, ds = 2.$$

By Theorem 3, we know $V_1[\tilde{\mathbf{x}}] \geq V_1[\mathbf{x}]$ with equality only if $\tilde{\mathbf{x}} = \mathbf{x}$. Changing variables, we find

$$V_1[\tilde{\mathbf{x}}] = \int_0^\ell \tilde{\eta} \, ds = \int_{-1}^1 \tilde{u}\sqrt{1 + \tilde{u}'^2} \, dx = V[\tilde{u}]$$

and

$$V_1[\mathbf{x}] = \int_0^\ell \eta \, ds = \int_{-1}^1 u\sqrt{1 + u'^2} \, dx = V[u].$$

The result evidently follows. □

Note 2. *The concave extremals are local maximizers.*

Chapter 3

Computation

3.1 Approximating the constant c

The key practical difficulty remaining in determining the shape of a hanging chain is finding an approximation for the solution of

$$c \sinh\left(\frac{1}{c}\right) = \frac{1}{2}\sqrt{\ell_1^2 - u_1^2}.$$

We recall the function $f(z) = \sinh z/z$ restricted to $z > 0$ has a well defined inverse as indicated in Figure 2.2, and if we can obtain a simple (and adequately accurate) approximation for the function $g(v) = f^{-1}(v)$ defined for $0 \leq v < \infty$, then a root find algorithm may be used to quickly determine the value of c. Let us briefly clarify the meaning of and need for a "simple and adequately accurate" approximation. A standard root find algorithm, like Newton's method or (to be more specific) Mathematica's `FindRoot` requires an **initial guess**. In order for the algorithm to return a value quickly and, in the presence of multiple solutions, a correct value, it is required that the initial guess be "simple" that is easy to calculate based on the value of v in terms of standard functions and "adequately accurate" so that one does not find extraneous roots. In the case at hand, in fact, the command

```
FindRoot[ Sinh[x]/x == v, {x, startingx(v)}]        (3.1)
```

where {`x`, `startingx(v)`} represents the directive to solve for `x` using the initial approximation `startingx(v)`, could produce a negative value if some care is not taken with the value of `startingx(v)`; this is because $f(x) =$

$\sinh x/x$ is an even function. We now describe how to obtain a simple and adequately accurate approximation for the value $g(v)$ where $g = f^{-1}$. We will use the following notation to distinguish among decimal approximation values. We write $f(a) \sim b$ to mean b is a rough approximation for $f(a)$, usually a simple and adequately accurate approximation. We use $f(z) \approx b$ to mean b is a numerically generated approximation accurate to a large (at least 5) decimal places.

We start with the observation that

$$\lim_{v \searrow 1} g'(v) = +\infty.$$

In order to get an accurate approximation for $g = f^{-1}$ near $v_0 = 1$, we compute the next derivative of f:

$$f''(z) = \frac{z^2 f_1' - 2z f_1}{z^4} = \frac{z^2 \sinh z - 2z \cosh z + 2 \sinh z}{z^3}.$$

Again, using L'Hopital's rule

$$\lim_{z \searrow 0} f''(z) = \lim_{z \searrow 0} \frac{\cosh z}{3} = \frac{1}{3}.$$

Thus, we have to leading order $v \sim 1 + g(v)^2/6$ or $g(v) \sim g_0(v) = \sqrt{6(v-1)}$. This provides an approximation with tolerance decaying to 0.01 around $v = 1.1$. See Figure 3.1.

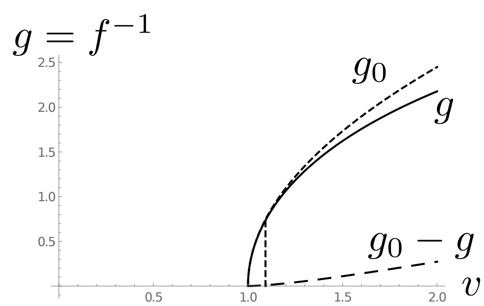

Figure 3.1: an initial approximation of $f^{-1}(v)$: The vertical line near $v = 1.1$ gives a tolerance for the approximation around 0.01.

This approach may be applied with center of expansion any $z > 0$ for a sequence of such values $z_1,\ z_2, \ldots$ In this way, we obtain a collection of approximations g_j each adequately accurate on some interval containing $\sinh(z_j)/z_j$. The Taylor approximation for f at $z = z_j$ gives

$$v = f(z) \sim f(z_j) + f'(z_j)(z - z_j) + \frac{f''(z_j)}{2}(z - z_j)^2.$$

Solving for $z \sim g(v)$ we have

$$g(v) \sim z_j + \frac{-f'(z_j) + \sqrt{f'(z_j)^2 - 2f''(z_j)(f(z_j) - v)}}{f''(z_j)}. \tag{3.2}$$

Taking the first such approximation centered at $v_1 = 1.1$ with $z_1 = \sinh(1.1)/1.1 \approx 1.21422$, we obtain an approximation $g_1(v)$ accurate to within 0.01 for $1.1 < v \le v_2 = 1.57$; see Figure 3.2.

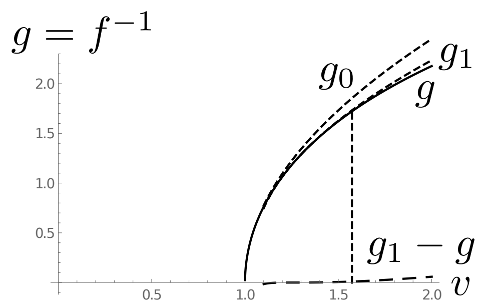

Figure 3.2: a second approximation of $f^{-1}(v)$: The vertical line near $v = 1.57$ gives a tolerance for the approximation around 0.01.

Before proceeding to further approximations in our sequence, we consider approximation of $g(v)$ when v is large.

3.2 The Lambert "W" function

There is a "standard" special function we can use to approximate $g(v) = f^{-1}(v)$ for large v. This is Lambert's "W" function known as "`ProductLog`" in Mathematica. Roughly speaking, W is the inverse of the complex valued function $h(z) = ze^z$ defined on $\mathbb{C}$. More precisely, W is defined on the Riemann surface $\mathcal{W}$ of h which we now partially describe. The basic formula for $h(z)$ is

$$h(x + iy) = e^x[x \cos y - y \sin y + i(y \cos y + x \sin y)].$$

The function $\xi(y) = -y \cot y$ is even and smooth when restricted to $-\pi < y < \pi$ (or any interval $k\pi < y < (k+1)\pi$). Also, ξ is increasing when restricted to $0 \le y < \pi$ and has an increasing inverse $\eta_0 : [0, \infty) \to [0, \pi)$ whose graph gives the upper boundary of the region Ω_0 on the left in Figure 3.3. The region

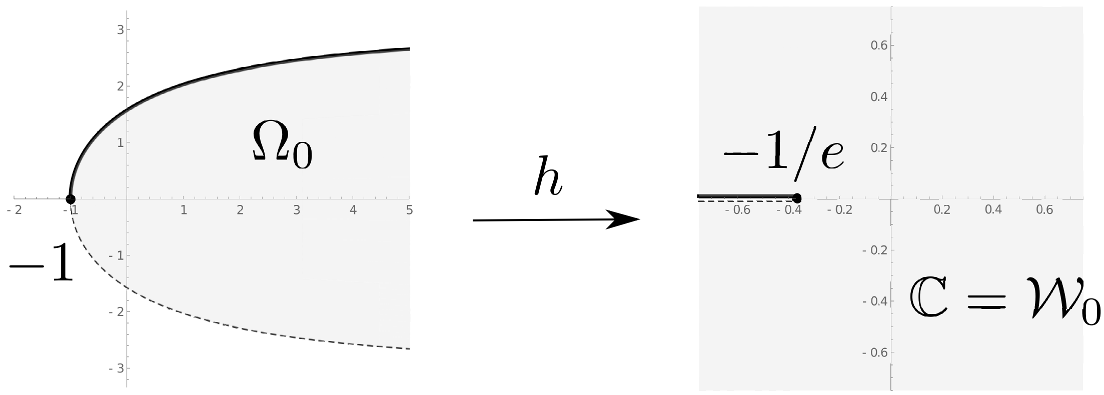

Figure 3.3: a fundamental domain for ze^z and its Riemann surface (initial sheet)

$$\Omega_0 = \{z = x + iy : -\eta_0(x) < y \le \eta_0(x)\}$$

is a fundamental domain for h with image $\mathcal{W}_0$ indicated on the right with a branch cut extending from $-1/e$ along the negative real axis. The inverse mapping $W_0 : \mathcal{W}_0 \to \mathbb{C}$ is the primary branch of the Lambert function. In particular, the function W_0 restricted to the real interval $[-1/e, \infty)$ gives the inverse of $h(x) = xe^x$ for $x \ge -1$. The graph of the restriction of h to the real line is indicated in Figure 3.4 Unfortunately, we need the inverse on the complementary interval $(-\infty, -1]$ which is not given by W_0. If we exit $\mathcal{W}_0$ along the image of the graph of η_0, which is the top of the branch cut along the negative real axis, we enter the region $\mathcal{W}_{-1}$ of the Riemann surface on which the inverse W_{-1} is defined. A convenient curve for reference within $\mathcal{W}_{-1}$ is obtained by considering again the function $\xi(y) = -y \cot y$ restricted to $\pi < y < 2\pi$. Since

$$\xi'(y) = -\cot y + y \csc^2 y = -\frac{1}{\sin y} \left(\cos y - \frac{y}{\sin y} \right),$$

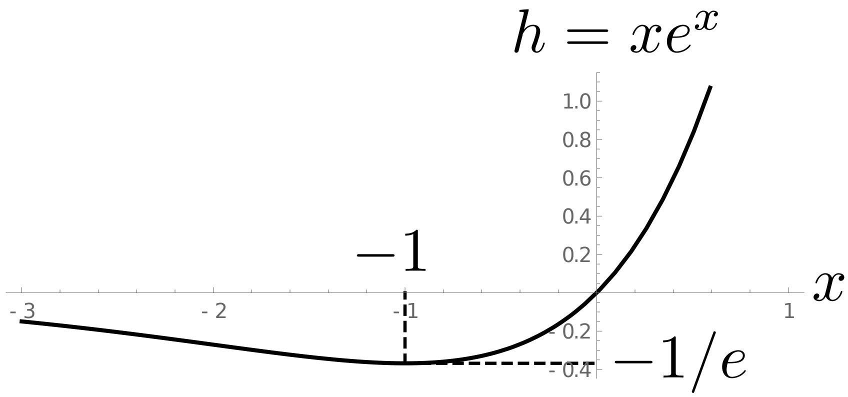

Figure 3.4: The restrictions to the intervals $(-\infty, -1]$ and $[-1, \infty)$ are invertible.

and $-1 < \sin y < 1$ while $y > \pi > 1$ on the interval $\pi < y < 2\pi$, we see ξ is increasing and takes all values in $\mathbb{R}$. Thus, this function has in inverse $\eta : \mathbb{R} \to (\pi, 2\pi)$ whose graph is indicated above the graph of η_0 in Figure 3.5. The region

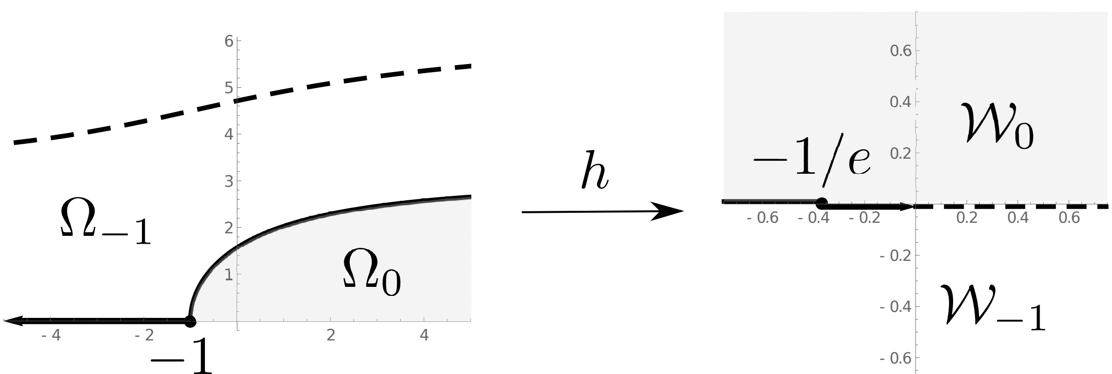

Figure 3.5: a portion of the fundamental region associated with W_{-1}

$$\Omega = \{z = x + iy : 0 < y \leq \eta(x)\} \cup \{x \in \mathbb{R} : x \leq -1/e\}$$

again covers the plane. The upper half plane in the image is part of $\mathcal{W}_0$ as before, but the lower half plane is now part of $\mathcal{W}_{-1}$. The image of the segment

$-\infty < x \leq -1$ along the negative real axis is the segment $-1/e \leq w < 0$ in $\mathcal{W}_{-1}$. Thus, the inverse of the restriction of $h(x) = xe^x$ to $(-\infty, -1]$ is given by $h^{-1}(\xi) = W_{-1}(\xi)$ for $-1/e \leq \xi < 0$.

Recall our function $g(v) = f^{-1}(v)$ where $f(z) = \sinh z/z$. When z is real and significantly greater than $z = 0$, the value of $f(z)$ is approximated by $e^z/(2z)$. Setting $\xi = -z$, this becomes

$$f(z) \sim -\frac{1}{2\xi e^{\xi}} = -\frac{1}{2h(\xi)}.$$

That is, if $v = f(z)$, then

$$\xi \sim h^{-1}\left(-\frac{1}{2v}\right) = W_{-1}\left(-\frac{1}{2v}\right),$$

and

$$g(v) = z \sim -h^{-1}\left(-\frac{1}{2v}\right) = -W_{-1}\left(-\frac{1}{2v}\right).$$

The approximation

$$g_{\infty}(v) = -W_{-1}\left(-\frac{1}{2v}\right)$$

gives an approximation of $g(v)$ within 0.01 for $v \geq 2.5$; see Figure 3.6.

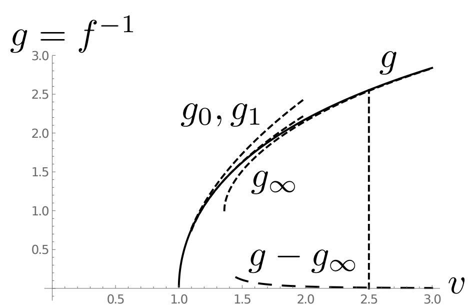

Figure 3.6: approximation of $f^{-1}(v)$ for v large: The vertical line near $v = 2.5$ gives a tolerance for the approximation around 0.01.

Thus, we see our sequence of approximations $g_j(v)$ has a termination point when we can approximate on the interval $[0, 2.5]$, or (given the approximations g_0 and g_1, when we can approximate for $1.57 < v < 2.5$). In fact, It is adequate to obtain approximations $g_2, \ldots, g_6$ corresponding to $v_2 = 1.57$, $v_3 = 1.78$, $v_4 = 1.93$, $v_5 = 2.07$, and $v_6 = 2.24$.

With the eight approximations, $g_0, \ldots, g_\infty$, we obtain a fast and reliable implementation of (3.1).

3.3 Alternative recursive approximation

The following was not used in our numerical solution of the equation $\sinh z/z = v$, but it may be of independent interest.

For $v > 1$, we also have recourse to a recursive approximation scheme. Picking an initial value g_0, for example, we could take $g_0 = \sinh^{-1}(v)$ which will be smaller than $g(v)$ as long as $v > \sinh(1)$. We see the actual value $g(v)$ satisfies

$$\frac{\sinh g(v)}{g(v)} = v \qquad \text{or} \qquad g(v) = \sinh^{-1}(vg(v)).$$

This suggests setting $g_1 = \sinh^{-1}(vg_0)$ and $g_{j+1} = \sinh^{-1}(vg_j)$ in general for $j = 0, 1, 2, \ldots$.

Conjecture 1. *The sequence g_j tends (upward) to $g(v)$ with the estimate*

$$g(v) - g_{j+1} \le g_{j+1} - g_j.$$

For example, if we take $v = 33.6189 \approx \sinh[6]/6$, then

$$\begin{aligned} g_0 &= \sinh^{-1}(v) \approx 4.20846 \\ g_1 &= \sinh^{-1}(g_0 v) \approx 5.64534 \\ g_2 &= \sinh^{-1}(g_1 v) \approx 5.93907 \\ g_3 &= \sinh^{-1}(g_2 v) \approx 5.98979. \end{aligned}$$

3.4 Numerical Code(s)

We begin with a first section in Mathematica obtaining a reliable numerical representation of the function g. As indicated in (3.2), expressions for f, f', and f'' are used in our initial approximations of g. With this in mind, we define

```
f[z_] = Sinh[z]/z;
fp[z_] = D[Sinh[z]/z, z];
fpp[z_] = D[Sinh[z]/z, z, 2];
```

The underscore appearing in `z_` indicates the declaration of a variable in Mathematica.

As suggested in (3.1) we begin with a general form of the function which remains unevaluted:

```
gGeneral[vinput_, vguess_] :=
FindRoot[Sinh[findz]/findz == vinput, {findz, vguess}][[1, 2]]
```

The cryptic `[[1,2]]` is required to extract the numerical value of the Mathematica output from `FindRoot` which has the form { `findz` → `value`}. Literally, "Take the first object within the vector (curly brackets) and take the second entry `value`."

We define our initial approximation near $v = 1$:

```
g0[w_] = Sqrt[6 (v-1)];
```

Setting our requirement for deviation from the actual inverse function g at $\Delta g = 0.01$, we determine the first of six approximation points as described in connection with Figure 3.1 above. The first value at which a new approximation is required is $v_1 = 1.1$. The others are determined iteratively and are given by $v_2 = 1.57$, $v_3 = 1.78$, $v_4 = 1.93$, $v_5 = 2.07$, and $v_6 = 2.24$ as mentioned above. The code determining v_1 is

```
maxvalidity0 =
FindRoot[
Sqrt[6(wfind - 1)] - gGeneral[wfind,Sqrt[6(wfind-1)]] == 0.01,
{wfind, 1.2}][[1, 2]]
> 1.09258
```

where we have indicated output by ">." It may also be noted that we have taken the initial guess in `gGeneral` to be the value of the first approximation g_0. The guess for `wfind` in this evaluation is made from the plot in Figure 3.1.

We then have six short subsections in which we determine the next six local approximations. These have the following form:

```
v1 = maxvalidity0
z1 = Sinh[1.1]/1.1
> 1.21422
g1[v_] =
z1 + (-fp[z1] + Sqrt[fp[z1]^2 - 2 fpp[z1] (f[z1] - v)])/fpp[z1]
```

```
maxvalidity1 =
FindRoot[g1[wfind] - gGeneral[wfind,g1[wfind]] == 0.01, {wfind,1.5}][[1,
2]]
> 1.57264
```

In each case, a plot similar to the one appearing in Figure 3.2 is required to make a rough guess for the last implementation of `RootFind` to determine the interval of validity (and the next point of evaluation). When these six subsections are complete, we define

```
gInfinity[v_] = -ProductLog[-1, -1/(2 v)]
```

from which we obtained

```
minvalidity =
FindRoot[gGeneral[wfind,g6[wfind]] - gInfinity[wfind] == 0.01,
{wfind, 2.3}][[1, 2]]
> 2.50092
```

Finally, we define a global approximation using the Mathematica function `Piecewise` used to define piecewise functions:

```
gGlobal[v_] = Piecewise[
Sqrt[6 (v - 1)], 1 < v < 1.1,
g1[v], 1.1 - 0.001 < v < 1.57,
g2[v], 1.57 - 0.001 < v < 1.78,
g3[v], 1.78 - 0.001 < v < 1.93,
g4[v], 1.93 - 0.001 < v < 2.07,
g5[v], 2.07 - 0.001 < v < 2.24,
g6[v], 2.24 - 0.001 < v < 2.5,
ginf[v], v > 2.5 - 0.001 ]
```

Finally, we complete this (main) section by setting

```
g[v_] := gGeneral[ v, gGlobal[v] ]
```

This completes the numerical approximation of $f(z) = \sinh(z)/z$.

When we have done this, equation (2.6) can be solved numerically and the relations (2.7) and (2.8) may be implemented to obtain a normalized chain shape defined for $-1 \leq x \leq 1$:

```
c = 1/g[Sqrt[ell^2 - u1^2]/2];
mu = -c ArcSinh[u1/Sqrt[ell^2 - u1^2]];
uNormalized[x_]=
c Cosh[(x - mu)/c] - c Cosh[(1 + mu)/c])
```

It remains to take specific physical coordinates, scale them to determine an appropriate ℓ and u_1, and then scale the shape determined above to fit the physical problem. We include the coding for this in the discussion of the experiment below.

Chapter 4

Experiment

4.1 Relation to physical parameters

Let us say we are given a chain of length $L > 0$ and two points (A, B) and (C, D) in the plane with

$$C > A \qquad \text{and} \qquad L^2 > (C - A)^2 + (D - B)^2.$$

In our code, we implement a test:

```
The following numbers should be positive:
C - A
L^2 - (C - A)^2 - (D - B)^2
```

In normalized coordinates we obtain

$$\ell = \frac{2}{C - A}\, L \qquad \text{and} \qquad u_1 = \frac{2}{C - A}\,(D - B).$$

Thus, if the test returns a positive result, we define

```
u1 = 2 (D - B)/(C - A);
ell = 2 L/(c - a);
```

We use these values to determine the parameters c and μ according to the analysis above. We have also the relation

$$X = A + \frac{C - A}{2}\,(x + 1) \qquad \text{or} \qquad x = -1 + \frac{2}{C - A}\,(X - A).$$

Substituting into the solution above, we obtain the physical chain shape is modeled by the graph of

$$U(X) = B + \frac{C-A}{2}\left[c\cosh\frac{1}{c}\left(\frac{2}{C-A}(X-A) - 1 - \mu\right) - c\cosh\left(\frac{1+\mu}{c}\right)\right].$$

with the obvious implementation.

4.2 Some specific shapes

On the next page we produce several chain shapes (produced for a chain of length 2 feet) which are scaled down to fit the page. One can take any small piece of chain and test the results by comparing to the shapes on this page.

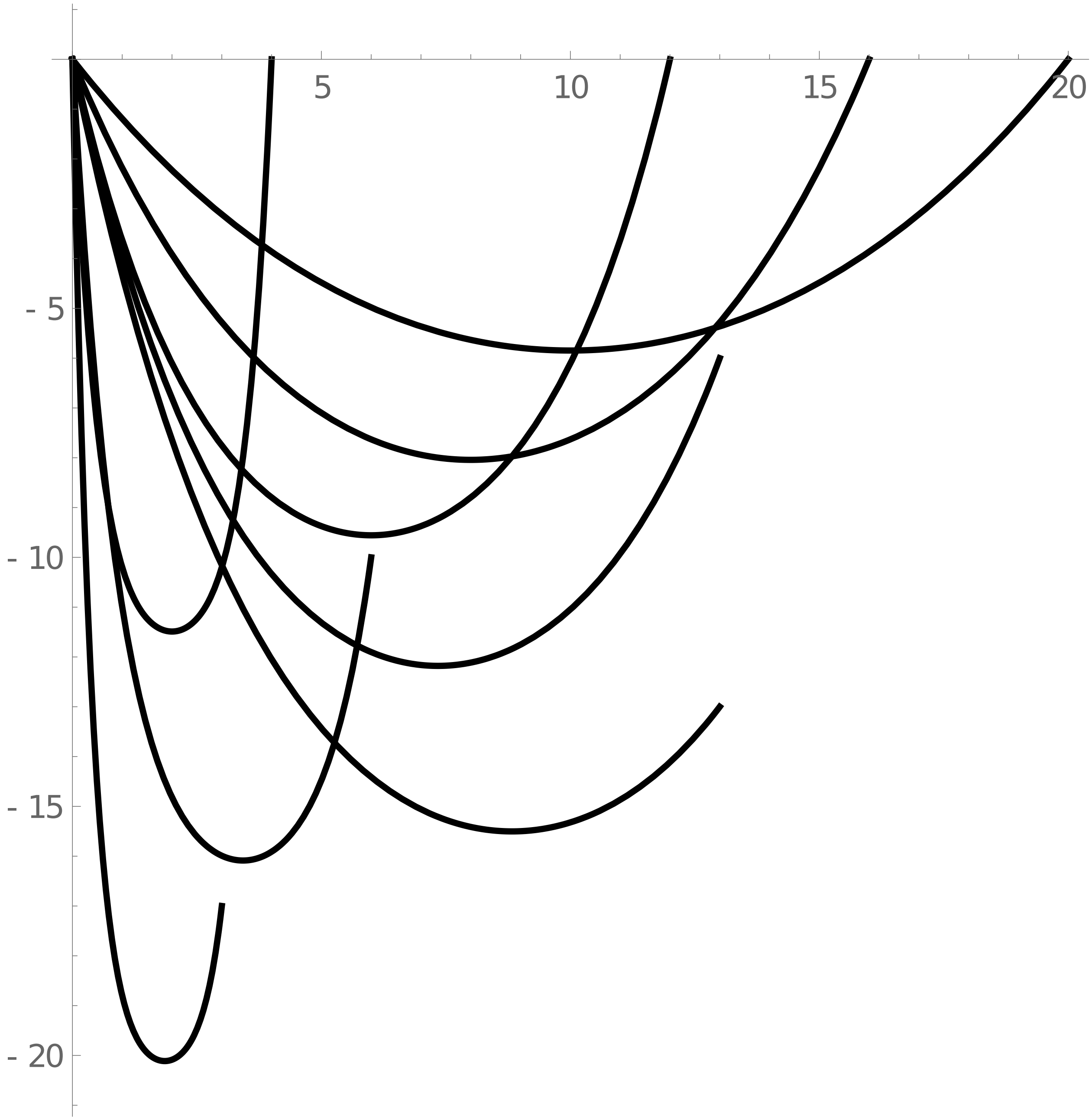

Figure 4.1: several chain shapes to test with physical chains

4.3 Error

Normally, we would include a section on error analysis here, but the physical chains agree with the predicted shapes so closely, that it is not clear how to measure any error.

4.4 Generalizations

There are at least three obvious generalizations for this problem and two more that I will list as well. Some have been done as projects before. I will list them in (roughly) the order of increasing difficulty. Each has something to do with changing the density of the chain or the flexibility of the chain.

1. If two lengths of chain of different densities are joined and the resulting single chain with a piecewise constant density hangs from two fixed points, then one can model the shape with a straightforward modification of the discussion above.

2. One could consider finitely many links, assumed to be straight line segments, hinged at the endpoints, and having prescribed lengths $\ell_1, \ell_2, \ldots, \ell_k$. This leads to a very interesting discrete minimization. Questions concerning approximation of shapes for the other problems is an obvious related question.

3. A direct generalization of the first problem above is to consider a general density $\rho = \rho(s)$ as a smooth function of arclength. This is a much more difficult problem. It is also somewhat difficult to easily obtain a large variety of subject chains for experimentation and modeling, though the next two problems suggest some possibilities.

4. One can consider attaching weights to various points on a hanging chain. If there are many links and the weights vary continuously, this can approximate some chains of chains of variable density.

5. One can include an elastic energy, which is an essentially different kind of problem. Nevertheless, a thin beam supported at both sides and supporting its own weight in gravity will sag/hang in an interesting shape which can be modeled.

Bibliography

[1] G. Buttazzo, M. Giaquinta, and S. Hildebrandt. *One-dimensional Variational Problems*. Oxford, 1998.

[2] J. Troutman. *Variational Calculus and Optimal Control, Second Edition*. Springer, 1983.