Calculus of Variations
Homework 1

January 11, 2012

1. If $\mathcal{F}$ is a functional and the first variation

$$\delta\mathcal{F}_u[\eta] = \lim_{h \to 0} \frac{\mathcal{F}[u + h\eta] - \mathcal{F}[u]}{h}$$

exists and u is a minimizer, show that $\delta\mathcal{F}_u[\eta]$ must be zero.

2. Show that $\mathcal{V} = \{v \in C^1[a,b] : v(a) = 0 = v(b)\}$ is a vector space.

3. Prove that if $u \in C^0[a,b]$ and $\int uv = 0$ for every $v \in \mathcal{V}$ (see previous problem), then $u \equiv 0$.

4. Find a function $\mu \in C_c^\infty[a,b]$ with $\int \mu = 1$.

5. Show that for $\mu \in C_c^\infty[a,b]$ the following are equivalent:

   (a) There is some $\eta \in C_c^\infty[a,b]$ such that $\mu = \eta'$.

   (b) $\int_a^b \mu(x)\,dx = 0$.

6. Prove the fundamental lemma of the calculus of variations: If $u \in C^0[a,b]$ and $\int_{[a,b]} u\eta = 0$ for all $\eta \in C_c^\infty[a,b]$, then $u \equiv 0$.