Calculus of Variations
Homework 4

February 13, 2012

1. Let
$$\mathcal{F}[u] = \int_a^b [u'^2 + q(x)u^2]\,dx$$
and consider minimization of $\mathcal{F}$ over
$$\{u \in C^1[a,b] : u(a) = 0 = u(b), \text{and } \int_a^b u^2\,dx = 1\}.$$
Let $u \in C^1[a,b]$ be a minimizer for this problem.

   (a) Show that there is some constant λ such that
   $$Lu = \lambda u,$$
   where $Lu = -u'' + qu$ is the associated Sturm-Liouville operator on $C^2[a,b]$.

   (b) There are, in fact, a sequence of eigenvalues $\lambda_1 < \lambda_1 < \cdots$ associated with the operator L, and the associated eigenfunctions form a complete orthonormal set in L^2. Show that $\lambda = \lambda_1$ is the least eigenvalue.

2. Assume that $t \mapsto y(t)$ for $t \in [0,1]$ parameterizes an embedded curve in $\{(x_1, x_2) : x_2 \geq 0\}$ with $y(0) = (-1,0)$ and $y(1) = (1,0)$. *Embeddedness* here means that the parameterization is one-to-one and smooth. The union of such a curve with the interval $\mathcal{I} = \{(x_1, 0) : -1 \leq x_1 <$

1} encloses a region Ω in the upper half plane. It follows from the divergence theorem that the area enclosed by the curve is

$$\int_\Omega 1 = \frac{1}{3}\int_0^1 (-y_1 y_2' + y_2 y_1')/|y'|\, dt.$$

What happened to the integral along the interval $\mathcal{I}$?

3. 6.3.1-3 (Sagan)