Notes on the calculus of variations

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Chapter 1

Introduction

These are some notes for a graduate course in the calculus of variations given at Georgia Tech in the Spring semester 2018. The primary text for the course is that of Buttazzo, Giaqinta, and Hildebrandt entitled *One-dimensional Variational Problems* (Oxford 1998). The text is at once more expansive than many texts on the calculus of variations, in that it covers topics in the direct methods and the requisite Sobolev and BV spaces, and is also limited in some ways omitting higher dimensional problems as well as some standard topics of practical importance; of particular note is the absence of exercises in the text. These notes aim to augment, supplement, (and in a certain sense) modify the presentation of the text.

The biggest additional component in the course is the detailed treatment of some specific applications of the calculus of variations to practical problems involving physical experiments as an integral part. In addition, we will mention higher dimensional variational problems briefly.

Finally, the text *Variational Calculus and Optimal Control* by John Troutman (Springer) will be used as a supplementary text. Some additional topics may be found there.

1.1 The basic problem

The basic problem of the course may be expressed in simple terms:

We wish to minimize the value of an integral

$$\int_a^b F(x, u(x), u'(x))\, dx$$

over some subset $\mathcal{A} \subset C^1[a,b]$*. That is, we wish to find a function* $u_0 \in \mathcal{A}$ *such that*

$$\int_a^b F(x, u_0(x), u_0'(x))\, dx \leq \int_a^b F(x, u(x), u'(x))\, dx$$

for every $u \in \mathcal{A}$*.*

While we will wish to consider various modifications or generalizations of this basic problem, we can start by thinking $-\infty < a < b < \infty$ and $F = F(x, z, p) : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function. This function, or sometimes the integrand $F(x, u(x), u'(x))$ obtained by plugging in a particular function $u \in C^1[a,b]$ into F, is called the **Lagrangian** or variational integrand.

Under these conditions, we have a well-defined function

$$\mathcal{F} : C^1[a,b] \to \mathbb{R} \quad \text{by} \quad \mathcal{F}[u] = \int_a^b F(x, u(x), u'(x))\, dx$$

where $C^1[a,b]$ denotes the collection of all continuously differentiable functions defined on the interval $[a,b]$.[1] A function like $\mathcal{F}$ (whose argument is a function and which takes real values) is sometimes called a functional. One that is defined in terms of an integral like $\mathcal{F}$ is called a variational integral or an **integral functional**. This course, roughly speaking, is about minimizing integral functionals.

The set $C^1[a,b]$ appearing in our formulation above may be considered as a superset or "universe" in which the minimization will take place. We may use different supersets, but this set will usually be a *linear space.* It should be noted, however, that the problem is not to minimize $\mathcal{F}$ with respect to all functions in the universe $C^1[a,b]$ but rather on some **admissible class** $\mathcal{A}$. The conditions imposed by $\mathcal{A}$ which define the *restriction*

$$\mathcal{F} : \mathcal{A} \to \mathbb{R} \quad \text{by} \quad \mathcal{F}[u] = \int_a^b F(x, u(x), u'(x))\, dx$$

are usually of major importance when it comes to the minimization problem posed above. For example, if we consider the Poisson integral

$$\mathcal{F}[u] = \int_0^1 [u + (u')^2]\, dx$$

[1]See Exercise 1.

over all of $C^1[0,1]$, then it is easy to check that the constant functions $u_n(x) \equiv -n$ satisfy $\mathcal{F}[u_n] = -n$, so there is no minimum. We shall see later that $\mathcal{F}$ has a finite minimum on $\mathcal{A} = \{u \in C^1[a,b] : u(a) = 0 = u(b)\}$.

One should not expect to solve the basic problem for any given Lagrangian. That is to say, some conditions should be satisfied. One can see this by simply negating the integrand in $\mathcal{F}$ above. Notice that each of the functions

$$u_n(x) = n\left[\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right]$$

belongs to $\mathcal{A} = \{u \in C^1[a,b] : u(a) = 0 = u(b)\}$, but if

$$\mathcal{G}[u] = \int_0^1 [-u - (u')^2]\, dx,$$

then

$$\mathcal{G}[u_n] = -\frac{n(2n-1)}{2} \to -\infty \quad \text{as} \quad n \to \infty.$$

A general principle which is worth remembering is the following:

> *Any difficulty which can arise in finite dimensional calculus can also arise in the calculus of variations.*

In making this comparison, the functional $\mathcal{F} : C^1[a,b] \to \mathbb{R}$ may be compared to a real valued function $f : \mathbb{R}^n \to \mathbb{R}$ with $\mathcal{A}$ playing the analogue of a particular subset $\Omega \subset \mathbb{R}^n$ on which the minimization is to take place.

The behavior we have observed for the Poisson integral is roughly analogous to the fact from one-dimensional calclulus that a continuous function $f : \mathbb{R} \to \mathbb{R}$ should not be expected to have a minimum on the entire real line, but will have a minimum on any compact interval $[a,b]$. The functions $u_n = -n$ may be thought of as "running off" to ∞.

The second example, with the negative of the Poisson integral, may be interpreted to demonstrate that there are multiple ways in which sets of functions can fail to be compact. Fixing the boundary values does not necessarily imply compactness. A sequence of functions can still "run off." Generally speaking, these functions can also be viewed as tending to the "boundary" of the universal set C^1 since the derivatives at the endpoints tend to ∞. (Though this example doesn't show it, this kind of phenomenon can also happen when the sequence of functions stays bounded.)

1.2 Standard examples; some generalizations

The following example may be considered as a simplification of the Poisson integral minimization mentioned above.

Example 1 *The* **Dirichlet Integral** *is given by*

$$\mathcal{D}[u] = \frac{1}{2}\int_a^b |u'|^2\,dx.$$

This functional is defined on all of $C^1[a,b]$*; a typical admissible class over which one may wish to minimize is*

$$\mathcal{A}_0 = \{u \in C^1[a,b] : u(a) = u_0,\ u(b) = u_1\}$$

where u_0 *and* u_1 *are fixed numbers specifying a so called Dirichlet boundary condition.*

A generalization one may wish to consider is minimization of the Dirichlet energy over the class

$$\mathcal{A}_1 = \{u \in C^0[a,b] \cap C^1(a,b) : u(a) = u_0,\ u(b) = u_1\}.$$

Notice that $\mathcal{A}_1 \not\subset C^1[a,b]$, however, the reverse inclusion holds, and it may be the case that a minimizer is, in fact, regular at the endpoints and in the space of higher regularity. It should also be observed that our concept of the values of $\mathcal{D}$ must be modified, for there are functions u in $\mathcal{A}_1$ for which $\mathcal{D}[u] = +\infty$. See Exercise 2.

Example 2 *The* **total variation** *of a function* $u \in C^1[a,b]$ *is given by*

$$\mathcal{T}[u] = \int_a^b |u'|\,dx.$$

This functional may also be minimized on $\mathcal{A}_0$ *or* $\mathcal{A}_1$*.*

Using $\mathcal{T}$ to construct a norm:

$$\|u\| = \int_a^b |u|\,dx + \mathcal{T}[u]$$

(called the "BV norm"), it is possible to extend the domain of $\mathcal{T}$ from $C^1[a,b]$ to the functions of **bounded variation**.

Example 3 *The length of the graph of u is given by*

$$\mathcal{L}[u] = \int_a^b \sqrt{1+(u')^2}\,dx.$$

Again, $\mathcal{A}_0$ and $\mathcal{A}_1$ are natural admissible classes for minimization.

Other admissible classes which are often convenient to consider are those of piecewise continuous or piecewise differentiable functions. Let us denote by $\sqsubset^0[a,b]$ the set of all functions u for which there is a partition $a = x_0 < x_1 < \cdots < x_k = b$ such that $u \in C^0[x_{j-1}, x_j]$ for $j = 1, \ldots, k$. By $\sqsubset^1[a,b]$ we mean the functions $u \in C^0[a,b]$ with derivatives $u' \in \sqsubset^0[a,b]$.

Other generalizations

Any of the standard examples above may be extended to vector valued functions of a single variable, that is to functions $u : [a,b] \to \mathbb{R}^m$, provided the vector valued function is in the appropriate space. In this case, the quantity $(u')^2$ should be replaced with the square of the norm of the velocity vector $|u'|^2$. The simplest space within which to find admissible classes is denoted by $C^1([a,b] \to \mathbb{R}^m)$.

A final obvious generalization involves real valued functions $u = u(\mathbf{x})$ of several variables $\mathbf{x} = (x_1, \ldots, x_n)$ and multiple integrals. In this case, the derivative should be interpreted as the gradient vector of partial derivatives $Du = (u_{x_1}, \ldots, u_{x_n})$. For example, if Ω is a smooth bounded domain in $\mathbb{R}^2$, then minimizing

$$A[u] = \int_\omega \sqrt{1+|Du|^2} \quad \text{over} \quad \mathcal{A} = \{u \in C^1(\bar{\Omega}) : u\big|_\Omega = u_0\},$$

where $\bar{\Omega}$ denotes the closure of Ω and u_0 are some fixed specified boundary values, leads to the classical problem of finding nonparametric *minimal surfaces*, i.e., the shapes of certain soap films that project simply onto a plane.

It is also possible to generalize the functionals above in both regards and consider admissible classes of functions $u : \Omega \to \mathbb{R}^m$ with $\Omega \subset \mathbb{R}^n$. Minimization of Dirichlet energy in this case leads to the subject of *harmonic maps*. These are also related to finding parametric minimal surfaces because the Dirichlet energy is equal to the area (or geometric mass) of the parametric image of the map $u : \Omega \to \mathbb{R}^m$ in special coordinates. But I'm getting ahead of myself...

Exercise 1 *Remember that a function is continuous at* $x \in [a, b]$ *if for any* $\epsilon > 0$*, there is some* $\delta > 0$ *such that* $|u(x+h) - u(x)| < \epsilon$ *whenever* $|h| < \delta$ *and* $x + h \in [a, b]$*. A derivative, on the other hand, is usually defined at points* x *in an open interval so that*

$$u'(x) = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h};$$

the value of h *can be either positive or negative. The function* u *is then* **continuously differentiable,** *or* $C^1(a, b)$*, if the derivative is continuous at each* x *in the open interval* (a, b)*. Technically, when we consider* $C^1[a, b]$ *on a closed interval, we should specify whether we mean functions which have an extension to a larger open interval* $(a - \epsilon, b + \epsilon)$ *and are* C^1 *there or if we mean functions in* $C^0[a, b] \cap C^1(a, b)$ *which also have one-sided derivatives*

$$\lim_{h \searrow 0} \frac{u(a+h) - u(x)}{h} \quad \text{and} \quad \lim_{h \nearrow 0} \frac{u(b+h) - u(x)}{h}$$

at a *and* b *and the function* $u'(x)$ *defined as the right derivative of* u *and* $x = a$*, the left derivative of* u *at* $x = b$ *and the two-sided derivative everywhere else, is continuous on* $[a, b]$*. Show these two notions of* $C^1[a, b]$ *are equivalent and show, in fact, any function in* $C^1[a, b]$ *has an extension to a function in* $C^1(\mathbb{R})$*.*

Exercise 2 *Consider the Dirichlet integral as a functional on the class* $\mathcal{A}_1$ *given above. Show that* $\mathcal{D} : \mathcal{A}_1 \to \mathbb{R} \cup \{+\infty\}$*. Assuming there is a function in* $\mathcal{A}_1$ *which minimizes* $\mathcal{D}$*, can you find (or guess) which one? If you have a guess, can you prove your guess is correct?*