Chapter 1

Indirect Methods

1.1 The first variation

Here we describe the main tool in the indirect methods of the calculus of variations. The basic strategy is to use what we know about interior minimization of functions of a single variable considered in elementary calculus, namely that the derivative must vanish at a minimum.

Before we get started we introduce a useful and interesting class of functions which play an important role in the calculus of variations—those which are smooth and have compact support. By smooth, we mean having derivatives of all orders which are also differentiable (and continuous). The term "compact support" is shorthand for "support compactly contained in the interior." In order to understand what we mean by this, let us first consider the notion of support. Roughly speaking, the *support* of a function is the set where the function is nonzero. More precisely, by **support**, we shall mean the closure of that set:

$$\operatorname{supp} u = \overline{\{x \in [a,b] : u(x) \neq 0\}}.$$

To say that the function u is compactly supported means $\operatorname{supp} u \subset (a,b)$.[1] We will denote this class with the suggestive symbolism $C_c^\infty(a,b)$ or simply C_c^∞ if the domain is understood. See Exercise 3.

Here is one more preperatory observation: The supersets, or universal sets, we have used in our discussion above, for example $C^1[a,b]$ or $\sqsubset^1[a,b]$

[1] More generally, we write $\operatorname{supp} u \subset\subset A$, where A is any set, to mean $\operatorname{supp} u$ is compact and contained in the interior of A. This is read "u has support compactly contained in (the interior of) A."

are *linear spaces*, but the admissible class $\mathcal{A}$ should not be expected to be a linear space. To be a linear space, in this context, means $au + bv$ is in the space whenever u and v are in the space and a and b are constants.

Let us assume we have an admissible class which shares the linear structure of the superset to the extent that

$$u + \epsilon\phi \in \mathcal{A} \quad \text{for all } \phi \in C_c^\infty(a, b) \text{ and } |\epsilon| \text{ small enough.}$$

In this case, the function $f(\epsilon) = \mathcal{F}(u + \epsilon\phi)$, assuming u and ϕ are fixed, is a smooth function of the single variable ϵ. One-dimensional calculus thus gives us a necessary condition for the function u to minimize $\mathcal{F}$.

Proposition 1 (Proposition 1.2 of BGH)**.** *Assume $\mathcal{F} : \mathcal{A} \to \mathbb{R}$, $u \in \mathcal{A} \subset C^1[a, b]$, and $F = F(x, z, p)$ satisfy the following*

1. *For each $\phi \in C_c^\infty(a, b)$, there is some epsilon$_0 > 0$ such that*

$$\mathcal{U}_\phi = \{u + \epsilon\phi : |\epsilon| < \epsilon_0\} \subset \mathcal{A},$$

2. *$\mathcal{F}[u] \le \mathcal{F}[\tilde{u}]$ for all $\tilde{u} \in \mathcal{A}$, and*

3. *$F \in C^1((a, b) \times \mathbb{R} \times \mathbb{R})$,*

then

$$\int_a^b [F_z(x, u, u')\phi + F_p(x, u, u')\phi'] \, dx = 0 \quad \textit{for all } \phi \in C_c^\infty(a, b). \tag{1.1}$$

Definition 1. *Any function $u \in C^1(a, b)$ satisfying (1.1) is called a* **weak extremal** *for $\mathcal{F}$.*

Note the Lagrangian F is a function of three real variables, and F_z and F_p denote partial derivatives of this function. Remember directional derivatives of a smooth function $F : \mathbb{R}^n \to \mathbb{R}$ may be defined by

$$\frac{\partial F}{\partial \mathbf{v}}(\mathbf{x}) = D_{\mathbf{v}} F(\mathbf{x}) = \lim_{h \to 0} \frac{F(\mathbf{x} + h\mathbf{v}) - F(\mathbf{x})}{h}.$$

Sometimes this definition is restricted to unit vectors. Partial derivatives are the special case when the unit vector is a standard coordinate vector; in the case of F, the second standard coordinate vector is $\mathbf{e}_2 = (0, 1, 0)$ and points along the positive z-axis resulting in the partial F_z.

With this definition of directional derivatives we have

$$D_{\mathbf{v}}F(\mathbf{x}) = DF(\mathbf{x}) \cdot \mathbf{v}$$

where DF is the gradient vector consisting of the partial derivatives. See Exercise 4. In the proof of Proposition 1, we will compute a derivative of the function $f(\epsilon) = \mathcal{F}(u + \epsilon\phi)$ with respect to ϵ to get the expression appearing in (1.1). Notice this derivative has the form

$$\lim_{h \to 0} \frac{\mathcal{F}[u + \epsilon\phi + h\phi] - \mathcal{F}[u + \epsilon\phi]}{h}$$

which bears a striking resemblance to the definition of a partial derivative with ϕ playing the role of the vector direction in which the derivative is taken. In fact, it will be shown that when we evaluate at $\epsilon = 0$, we are calculating

$$\delta\mathcal{F}_u[\phi] = \lim_{h \to 0} \frac{\mathcal{F}[u + h\phi] - \mathcal{F}[u]}{h}.$$

This is called the **first variation**. Recall that $dF_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^n \to \mathbb{R}$. We have $\delta\mathcal{F}_u : C_c^\infty(a, b) \to \mathbb{R}$. In this sense, $C_c^\infty(a, b)$ represents the tangent space to $C^1[a, b]$ at u.

Proof that a minimizer is a weak extremal: If ϵ is small enough, say $|\epsilon| < \epsilon_0$, then $u_0 + \epsilon\phi \in \mathcal{A}$. Thus, we can define the real valued function $f : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ of the single variable ϵ by

$$f(\epsilon) = \mathcal{F}[u_0 + \epsilon\phi],$$

and (for ϕ fixed) we have $f(0) \leq f(\epsilon)$. It follows from one-dimensional calculus that $f'(0) = 0$. That is,

$$\frac{d}{d\epsilon} \int_a^b F(x, u_0 + \epsilon\phi, u_0' + \epsilon\phi')\, dx\Big|_{\epsilon=0} = 0.$$

Differentiating under the integral sign, we get

$$\int_a^b \frac{d}{d\epsilon} F(x, u_0 + \epsilon\phi, u_0' + \epsilon\phi')\, dx\Big|_{\epsilon=0} = 0.$$

Thus, we can obtain (1.1) from the chain rule. It will also be noted that the same expression is obtained using the mean value theorem for functions of several variables and taking the limit directly. That is,

$$\begin{aligned}\lim_{h\to 0}\frac{\mathcal{F}[u+h\phi]-\mathcal{F}[u]}{h} &= \lim_{h\to 0}\int_a^b \frac{F(x,u+h\phi,u'+h\phi')-f(x,u,u')}{h}\,dx \\ &= \lim_{h\to 0}\int_a^b [F_z(x,u+h^*\phi,u'+h^*\phi')\phi \\ &\qquad - F_p(x,u+h^*\phi,u'+h^*\phi')\phi']\,dx\end{aligned}$$

where h^* is some number between 0 and h. Therefore, h^* tends to zero with h, and

$$\delta\mathcal{F}_u[\phi] = \lim_{h\to 0}\frac{\mathcal{F}[u+h\phi]-\mathcal{F}[u]}{h}$$

as claimed above. □

We conclude our discussion of Proposition 1 with a formal summary of our discussion of the first variation:

Definition 2. *Given a C^1 Lagrangian, the* **first variation** *of*

$$\mathcal{F}[u] = \int_a^b F(x,u,u')\,dx$$

at $u \in C^1[a,b]$ is the functional $\delta\mathcal{F}_u : C_c^\infty(a,b) \to \mathbb{R}$ by

$$\delta\mathcal{F}_u[\phi] = \frac{d}{d\epsilon}\mathcal{F}(u+\epsilon\phi)\Big|_{\epsilon=0} = \lim_{h\to 0}\frac{\mathcal{F}[u+h\phi]-\mathcal{F}[u]}{h}.$$

This should be interpreted as the derivative of $\mathcal{F}$ at u in the direction ϕ.

The fundamental lemma

In order better understand the properties of certain weak extremals, we need to know what integral information like (1.1) implies about the function u. We begin with a relatively simple observation:

Lemma 1 (Lemma 1.3 in BGH)**.** *(fundamental lemma of the calculus of variations) If $f \in C^0(a,b)$ satisfies*

$$\int_a^b f(x)\eta(x)\,dx = 0 \quad \textit{for all } \eta \in C_c^\infty(a,b), \tag{1.2}$$

then $f(x) \equiv 0$ on (a,b).

Proof: Let $x_0 \in (a, b)$. It is enough to show $f(x_0) = 0$.
Let $\phi \in C_c^\infty(-1, 1)$ with $\phi \geq 0$ and

$$\int_{-1}^{1} \phi(x)\, dx = 1.$$

By extension, we can consider $\phi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \phi \subset\subset (-1, 1)$ and $\int_{\mathbb{R}} \phi = 1$. Next, we define a family of C_c^∞ functions indexed by $\delta > 0$:

$$\mu = \mu_\delta(x) = \frac{1}{\delta} \phi\left(\frac{x}{\delta}\right). \tag{1.3}$$

One can check that $\mu \in C_c^\infty(-\delta, \delta)$ is nonnegative and has $\int \mu \equiv 1$. See Exercise 6. The family $\{\mu = \mu_\delta\}$ is called the "standard" **mollifier** sequence or an **approximate identity**. We claim

$$f(x_0) = \lim_{\delta \to 0} \int_a^b f(x) \mu_\delta(x - x_0)\, dx. \tag{1.4}$$

To see this, let $\epsilon > 0$ and note that by continuity there is some $\delta > 0$ such that

$$|x - x_0| < \delta \qquad \text{implies} \qquad |f(x) - f(x_0)| < \epsilon.$$

In particular, we have

$$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon,$$

so

$$[f(x_0) - \epsilon] \int \eta \leq \int f\eta \leq [f(x_0) + \epsilon] \int \eta$$

where $\eta(x) = \mu(x - x_0)$. On the other hand, when δ is small, one can check that $\eta \in C_c^\infty(x_0 - \delta, x_0 + \delta) \subset C_c^\infty(a, b)$ with $\int \eta = 1$. Thus, if δ is small enough

$$f(x_0) - \epsilon \leq \int f\eta = \int_a^b f(x)\mu_\delta(x - x_0)\, dx \leq f(x_0) + \epsilon,$$

and (1.4) is established. But this also means

$$f(x_0) = \lim_{\delta \to 0} \int_a^b f(x) \mu_\delta(x - x_0)\, dx = \lim_{\delta \to 0} \int f\eta = 0. \qquad \square$$

Another proof of the fundamental lemma

In BGH the authors make, essentially, a specific choice of approximate identity:

$$\bar{\mu}(x) = \frac{1}{2\delta}\chi_{(-\delta,\delta)}$$

where

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

represents the characteristic function of the set A. It will be noted that this function is not in the class C_c^∞ used in (1.2) and used, in general, to define an approximate identity. In fact, the characteristic function is not even continuous. Nevertheless, it is possible to approximate $\chi_{(-\delta,\delta)}$ with smooth functions in order to conclude

$$\int_a^b f(x)\bar{\mu}(x - x_0)\, dx = 0. \tag{1.5}$$

The authors express this approximation process by saying $C_c^\infty(a, b)$ is "dense" in $L^2(a, b)$. Let us briefly describe the approximation and how one arrives at (1.5). We begin with the smooth approximate identity

$$\mu_\epsilon(x) = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right)$$

from (1.3) and form the **convolution integral** or molification

$$\mu_\epsilon * \bar{\mu}(x) = \int_{-\infty}^{\infty} \mu(\xi)\bar{\mu}(\xi - x)\, d\xi.$$

The function $\bar{\mu}$, though not continuous, is in L^p for every $p > 0$. It may be recalled that L^p for $0 < p < \infty$ is the set of measurable functions f for which the integral

$$\int |f|^p < \infty,$$

i.e., the set of functions, the absolute value of which has p-th power integrable. Each of the function spaces L^p is a vector space with a norm given by

$$|f|_{L^2} = \left(\int |f|^p\right)^{1/p}.$$

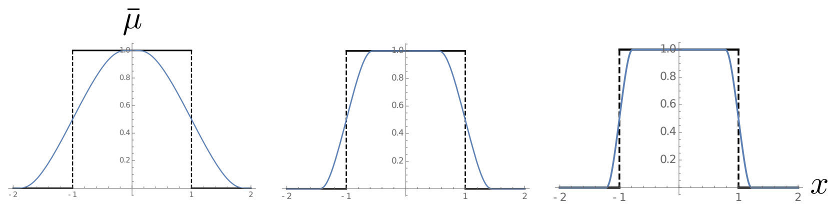

Figure 1.1: Here we indicate the approximation of a characteristic function by convolution with a standard mollifier. This process is called mollification and may be applied to a variety of functions.

Each of these spaces is complete under the norm, and in the special case $p = 2$, the norm is given by an inner product

$$\langle f, g\rangle = \int fg$$

satisfying the Cauchy-Schwarz inequality:

$$|\langle f, g\rangle| \leq |f|_{L^2}|g|_{L^2}.$$

It can be shown, see Exercise 8, that $\mu_\epsilon * \bar{\mu}$ approximates $\bar{\mu}$ as indicated in Figure 1.1. In particular,

$$\lim_{\epsilon\to 0} |\mu_\epsilon * \bar{\mu} - \bar{\mu}|_{L^p} = 0. \tag{1.6}$$

Thus, given any $f \in L^2(a, b)$

$$\begin{aligned}
\left|\int_{x\in(a,b)} f(x)\mu_\epsilon * \bar{\mu}(x - x_0) - \int_{x\in(a,b)} f(x)\bar{\mu}(x - x_0)\right| \\
&\leq \int_{x\in(a,b)} |f(x)||\mu_\epsilon * \bar{\mu}(x - x_0) - \bar{\mu}(x - x_0)| \\
&= \langle |f|, |\mathrm{shift}_{x_0}[\mu_\epsilon * \bar{\mu} - \bar{\mu}]|\rangle_{L^2(a,b)} \\
&\leq |f|_{L^2}|\mu_\epsilon * \bar{\mu} - \bar{\mu}]|_{L^2}
\end{aligned}$$

where $\mathrm{shift}_{x_0}[g](x) = g(x - x_0)$. In view of (1.6) we have

$$\lim_{\epsilon\to 0} \int_{x\in(a,b)} f(x)\mu_\epsilon * \bar{\mu}(x - x_0) = \int_{x\in(a,b)} f(x)\bar{\mu}(x - x_0).$$

In the context of the fundamental lemma, we have not only $f \in L^2(a,b)$ but $f \in C^0[a,b]$, and we know

$$\int_a^b f(x)\eta(x)\,dx = 0 \quad \text{for all } \eta \in C_c^\infty(a,b).$$

According to Exercise 8 we also have $\mu_\epsilon * \bar{\mu} \in C_c^\infty(-\delta,\delta)$ when ϵ is small, so when $x_0 \in (a,b)$ and ϵ is small, $\text{shift}_{x_0}[\mu_\epsilon * \bar{\mu}] \in C_c^\infty(a,b)$ and we can write

$$\begin{aligned}\frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta} f(x)\,dx &= \int_a^b f(x)\bar{\mu}(x-x_0)\,dx \\ &= \lim_{\epsilon\to 0}\int_a^b f(x)\mu_\epsilon * \bar{\mu}(x-x_0)\,dx \\ &= 0.\end{aligned}$$

This establishes assertion (1.10) of BGH that

$$\frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta} f(x)\,dx = 0,$$

and one can proceed with a simplified version of the reasoning given in the first proof:

$$f(x_0) - \epsilon \le \frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta} f(x)\,dx = 0 \le f(x_0) + \epsilon$$

whence

$$-\epsilon \le f(x_0) \le \epsilon. \qquad \square$$

Finally, we may note

$$\frac{1}{2\delta}\int_{(x_0-\delta,x_0+\delta)} g$$

for a general integrable function g is the average value of g over the interval $(x_0-\delta, x_0+\delta)$. This interpretation leads to a more general result:

Lemma 2 (Lemma 1.4 in BGH)**.** *(fundamental lemma with weaker regularity) If $f \in L^1_{\text{loc}}(a,b)$ satisfies*

$$\int_{(a,b)} f\eta = 0 \quad \textit{for all } \eta \in C_c^\infty(a,b), \tag{1.7}$$

then $f \equiv 0$, i.e., $f = 0$ almost everywhere, or more precisely $f(x) = 0$ for each Lebesgue point in (a,b).

Before we give the proof, let us give some account of the space L^1_{loc} and Lebesgue points. A measurable function f is said to be in $L^1_{\text{loc}}(a,b)$ if

$$\int_{(a',b')} |f| < \infty$$

whenever $(a',b') \subset\subset (a,b)$. Measurable "functions" are not defined pointwise but rather as equivalence classes in terms of integration (or more properly measure). Nevertheless, each such class of functions determines a kind of continuity:

Theorem 1 (Lebesgue's continuity theorem[2])**.** *If $f \in [f_0] \in L^1_{\text{loc}}(a,b)$, then for almost every $x \in (a,b)$,*

$$\lim_{\delta \searrow 0} \frac{1}{2\delta} \int_{(x-\delta,x+\delta)} f = f_0(x).$$

The assertion includes the fact that the limit exists. In fact, the stronger assertion

$$\lim_{\delta \searrow 0} \frac{1}{2\delta} \int_{\xi \in (x-\delta,x+\delta)} |f(\xi) - f(x)| = 0 \quad \textit{for almost every } x \in (a,b) \tag{1.8}$$

also holds. A point x for which (1.8) holds is called a **Lebesgue point** *of f.*

Lebesgue's functions

We've mentioned that "functions" in L^p are considered as "equivalence classes" of functions rather than individual functions defined pointwise. Since this point of view may be unfamiliar to some, let me at least try to give some heuristic explanation to indicate what this means.

Let's start with the simplest function in $C^0[0,1]$, namely $f(x) \equiv 0$. There is a "different" function defined by

$$f_1(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0. \end{cases}$$

This function is not very different from f. It is not continuous, but from the point of view of integration, all intgrals of the form

$$\int_0^1 g(x) f_1(x)\, dx,$$

[2]Sometimes this result is called Lebesgue's differentiation theorem.

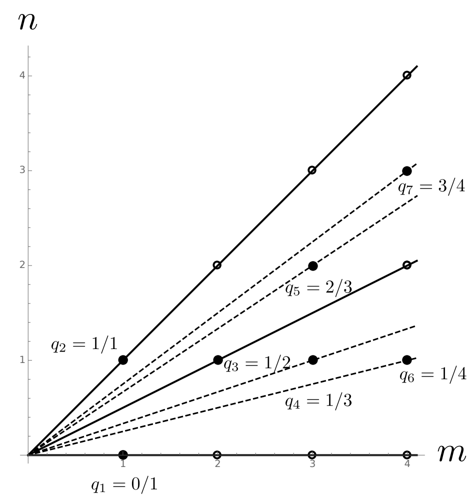

Figure 1.2: Here is a plot of the first seven rational numbers represented as ordered pairs of positive integers.

where g is some function we can integrate, are well defined and have the same value if we use f or f_1, namely zero. In fact, the rational numbers between 0 and 1 can be listed off, and if we change the value of f on the first k of them to obtain a function f_k, then we obtain a function which, from the point of view of integration, are just like (i.e., essentially indistinguishable) from f_1 or f. Let's carry out this construction so we can see how it looks visually. The listing of the rationals in $[0, 1]$ is

$$\mathbb{Q} \cap [0,1] = \left\{1, 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots\right\}.$$

That is, there is a one-to-one correspondence between $\mathbb{Q} \cap [0, 1]$ and the ordered pairs of integers (m, n) with the ordered pair corresponding to n/m so that $m > 0$, $m \geq n \geq 0$, and each point (m, n) produces a line of deleted multiples (km, kn) for $k = 2, 3, 4, \dots$. After deletions, we use a dictionary style ordering with $(p, q) < (m, n)$ if $p < m$ (or if $p = m$ and $q < n$). See Figure 1.2. Then we just list off the rational numbers in order $q_1, q_2, q_3, \dots$. We define for each $k = 1, 2, 3, \dots$

$$f_k(x) = \begin{cases} 1, & x = q_1, q_2, \dots, q_k \\ 0, & x \neq q_1, q_2, \dots, q_k, \end{cases}$$

and the function f_k has

$$\int_0^1 g(x)f_k(x)\,dx = \int_0^1 g(x)f(x)\,dx = 0 \quad \text{for all } g \in C^0[0,1].$$

It occured to Lebesgue that there was nothing particularly special about the rational numbers. You could change the values of f to anything you like on any finite set, and you still get the same integrals. This tells you that, from the point of view of integration, the value of f at any particular point, doesn't really matter. But this sounds like nonsense. Obviously, the value of the function must matter. In view of this paradox, Lebesgue changed

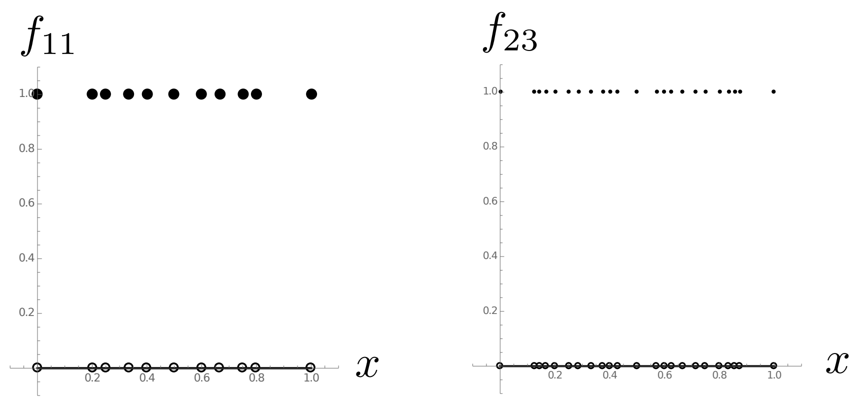

Figure 1.3: Here are plots of f_{11} which is the zero function with the value at the first 11 rational numbers changed to 1 and f_{23} which is the zero function with the value at the first 23 rational numbers changed to 1. From the point of view of integration, these functions are indistinguishable from the zero function.

his notion of a function. Instead of considering pointwise values alone, he decided to let

$$[0] = \{f : \int gf = 0 \text{ for all } g\}.$$

This is the *equivalence class* of all functions which are indistinguishable from the zero function in the sense of integration. Lebesgue went on to make a careful study of all the kinds of sets upon which it might make sense to change the values of any one representative in the class but get the same integrals. That property became known as *having measure* 0.

Proof of the fundamental lemma with weaker regularity:

The basic strategy of this proof is similar to the one used in BGH to prove the fundamental lemma when f is continuous. That is, we wish to show

$$\int_{(a,b)} f\chi_{(x_0-\delta,x_0+\delta)} = 0 \tag{1.9}$$

when $(x_0-\delta, x_0+\delta) \subset\subset (a,b)$. This is essentially (1.5). Our previous estimation of

$$\left| \int_{(a,b)} f\mu_\epsilon * \chi_{(x_0-\delta,x_0+\delta)} - \int_{(a,b)} f\chi_{(x_0-\delta,x_0+\delta)} \right| \tag{1.10}$$

using the Cauchy-Schwarz inequality does not go through as it stands because we no longer necessarily have $f \in L^2$. Following BGH we let $\delta > 0$ be fixed so that $[x_0-\delta, x_0+\delta] \subset\subset (a,b)$ and consider $\bar\mu = \bar\mu_{\bar\delta} \in \sqsubset^1_c(a,b)$ given by

$$\bar\mu(x) = \begin{cases} -\frac{1}{2\bar\delta}(|x-x_0| - \delta - \bar\delta) & \text{if } \delta - \bar\delta \le |x-x_0| \le \delta + \bar\delta, \text{ and} \\ \chi_{(x_0-\delta,x_0+\delta)} & \text{otherwise.} \end{cases}$$

where $\bar\delta < \delta$ is small enough so $\operatorname{supp}(\bar\mu) \subset\subset (a,b)$. The function $\bar\mu$ is a piecewise affine function as indicated in Figure 1.4. It is now argued that

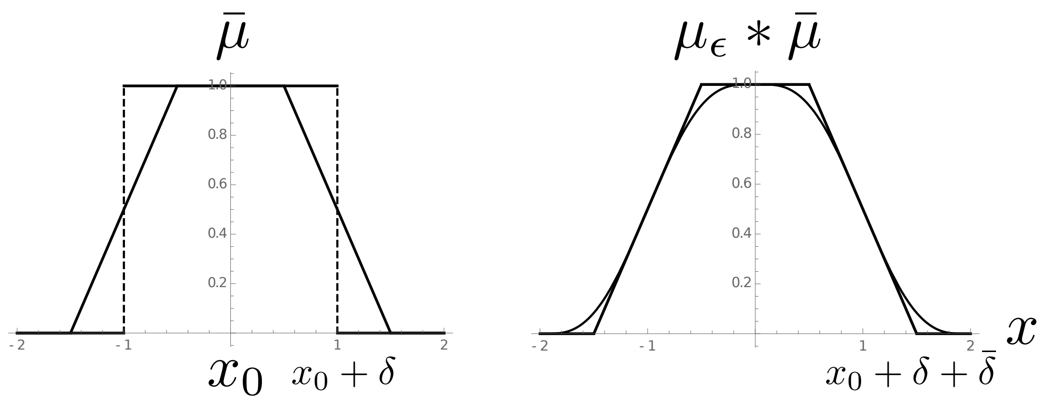

Figure 1.4: a piecewise continuous approximation of a characteristic function and its mollification

$C^\infty_c(a,b)$ is dense in $C^0[a,b]$ so that

$$\int_{(a,b)} f\bar\mu = \int_{(a,b)} f\bar\mu_{\bar\delta} = 0. \tag{1.11}$$

The norm on C^0 is the sup norm

$$\|u\| = \sup_{x \in [a,b]} |u(x)|$$

so that the distance between continuous functions u and v is $\sup|u - v|$. The sup norm is also called the *uniform norm* or the L^∞ norm, and the associated metric is called the *metric of uniform convergence*. Let us consider carefully this approximation procedure. Again, we use the approximate identity μ_ϵ from (1.3) and molify $\bar{\mu}$:

$$\mu_\epsilon * \bar{\mu}(x) = \int_\xi \mu_\epsilon(\xi)\bar{\mu}(x - \xi).$$

It follows from the reasoning of Exercise 8 that when ϵ is small enough, we have $\mu_\epsilon * \bar{\mu} \in C_c^\infty(a, b)$. Let us be specific about this by choosing a fixed ϵ_0 so that

$$\operatorname{supp}(\mu_\epsilon * \bar{\mu}) \subset \operatorname{supp}(\mu_{\epsilon_0} * \bar{\mu}) \subset\subset (a, b) \quad \text{for} \quad 0 < \epsilon < \epsilon_0.$$

Also, it is easy to check that for any x and $\tilde{x}$

$$|\bar{\mu}(\tilde{x}) - \bar{\mu}(x)| < \frac{1}{2\bar{\delta}}|\tilde{x} - x|. \tag{1.12}$$

This is a statement of *Lipschitz continuity*, and it can be shown that any function in $\sqsubset^1[a, b]$ is Lipschitz continuous. See Exercise 9. In particular, we may take $\alpha = \epsilon$ so that

$$|\tilde{x} - x| < \epsilon \qquad \Rightarrow \qquad |\bar{\mu}(\tilde{x}) - \bar{\mu}(x)| < \frac{\epsilon}{2\bar{\delta}}. \tag{1.13}$$

We can then estimate as follows:

$$\begin{aligned} \|\mu_\epsilon * \bar{\mu} - \bar{\mu}\|_{C^0[a,b]} &= \sup_{x \in \mathbb{R}} \left| \int_{\xi \in (-\epsilon, \epsilon)} \mu_\epsilon(\xi)[\bar{\mu}(x - \xi) - \bar{\mu}(x)] \right| \\ &\leq \sup_{x \in \mathbb{R}} \int_{\xi \in (-\epsilon, \epsilon)} \mu_\epsilon(\xi)|\bar{\mu}(x - \xi) - \bar{\mu}(x)| \\ &\leq \frac{\epsilon}{2\bar{\delta}}. \end{aligned}$$

It follows that

$$\begin{aligned}\left|\int_{(a,b)} f\mu_\epsilon * \bar{\mu} - \int_{(a,b)} f\bar{\mu}\right| &\leq \int_{\operatorname{supp}(\mu_{\epsilon_0}*\bar{\mu})} |f||\mu_\epsilon * \bar{\mu} - \bar{\mu}| \\ &\leq \frac{\epsilon}{2\bar{\delta}} \|f\|_{L^1(\operatorname{supp}(\mu_{\epsilon_0}*\bar{\mu}))} \\ &\to 0 \qquad \text{as } \epsilon \searrow 0.\end{aligned}$$

Since $\mu_\epsilon * \bar{\mu} \in C_c^\infty(a,b)$, we have shown

$$\int_{(a,b)} f\bar{\mu} = \lim_{\epsilon\searrow 0} \int_{(a,b)} f\mu_\epsilon * \bar{\mu} = 0.$$

This is (1.11). We do not have $\bar{\mu} = \bar{\mu}_{\bar{\delta}}$ converging uniformly to $\chi_{(x_0-\delta,x_0+\delta)}$ as $\bar{\delta} \searrow 0$. For this reason we need another device to show the integral in (1.11) approximates that in (1.9) and conclude

$$\lim_{\bar{\delta}\searrow 0} \int f\bar{\mu}_{\bar{\delta}} = \int_{(x_0-\delta,x_0+\delta)} f.$$

The appropriate result is the *Lebesgue dominated convergence theorem*[3] (DCT):

Theorem 2. *If*

1. *A sequence of functions $g_1, g_2, g_3, \ldots$ in $L^1(A)$ converges pointwise at almost every point in A to a measureable function g, i.e., the set of points where the sequence does not converge to g has measure zero, and*

2. *There is a function $G \in L^1(A)$ such that $|g_j| \leq G$ for $j = 1, 2, 3, \ldots$,*

then $g \in L^1(A)$ and

$$\lim_{j\to\infty} \int_A g_j = \int_A g.$$

It is required, then, that we have a pointwise limit (almost everywhere) and a dominating function.

The reasoning of Exercise 8 gives $|\bar{\mu}| \leq 1$ for all $\bar{\delta}$. To be explicit, let us fix δ and $\bar{\delta}_0$ so that

$$\operatorname{supp}(\bar{\mu}_{\bar{\delta}}) \subset \operatorname{supp}(\bar{\mu}_{\bar{\delta}_0}) \subset\subset (a,b) \quad \text{for} \quad 0 < \bar{\delta} < \bar{\delta}_0.$$

[3]Lebesgue's dominated convergence theorem is Theorem 4.4.15 in Royden's *Real Analysis* and is Theorem 1.34 in Rudin's *Real and Complex Analysis*.

Then we can take the dominating function to be

$$G = |f\chi_{\mathrm{supp}(\bar{\mu}_{\bar{\delta}_0})}|$$

so that

$$|f\bar{\mu}| = |f\bar{\mu}_{\bar{\delta}}| \leq G \quad \text{and} \quad \|G\|_{L^1(a,b)} = \|f\|_{L^1(\mathrm{supp}(\bar{\mu}_{\bar{\delta}_0}))} < \infty.$$

Finally, $f\bar{\mu} = f\bar{\mu}_{\bar{\delta}}$ converges pointwise to $f\chi_{(x_0-\delta,x_0+\delta)}$ at every point in (a,b) except the two points $x_0 \pm \delta$. Therefore, by the DCT

$$\int_{(x_0-\delta,x_0+\delta)} f = \lim_{\bar{\delta}\searrow 0} \int_{(a,b)} f\bar{\mu}_{\bar{\delta}} = 0.$$

This is (1.9). Now we can take x_0 to be a Lebesgue point, and we have

$$f(x_0) = \lim_{\delta\searrow 0} \int_{(x_0-\delta,x_0+\delta)} f = 0. \qquad \square$$

Exercises

Exercise 3. *Consider a function $\phi \in C_c^\infty(a,b)$.*

1. *Let $x_0 \in \partial\,\mathrm{supp}(\phi)$. Find the Taylor series for ϕ at x_0.*
2. *Find a function $\phi \in C_c^\infty(-1,1)$ with $\phi(0) = 1$.*

Exercise 4. *What is $T_{\mathbf{x}}\mathbb{R}^n$? Show that if one restricts the linear map $du_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^n \to \mathbb{R}$ to unit vectors $\mathbf{v}$, then the maximum value occurs for $\mathbf{v} = Du(x)/|Du(x)|$. What is the maximum value?*

Exercise 5. *The natural domain for integrals $\int f\phi$, like the one appearing in (1.2), is the collection of* **locally integrable functions**. *The function $f(x) = 1/(x-a)$ has $f \in C^0(a,b)$ but*

$$\int_a^b f(x)\,dx = +\infty.$$

Why does the integral $\int f\phi$ for $\phi \in C_c^\infty(a,b)$ still make sense for such a function? What is the definition of $L^1_{\mathrm{loc}}(a,b)$? Notice that the same considerations apply to (1.1).

Exercise 6. *Recall that the standard mollifier is defined by*

$$\mu(x) = \frac{1}{\delta}\phi\left(\frac{x}{\delta}\right),$$

where $\phi \in C_c^\infty(-1,1)$ *is nonnegative, fixed, and* $\int \phi = 1$.

1. *Show* $\mu \in C_c^\infty(-\delta,\delta)$ *with* $\int \mu \equiv 1$.
2. *Show* $\eta(x) = \mu(x - x_0)$ *has* $\eta \in C_c^\infty(x_0 - \delta, x_0 + \delta)$ *with* $\int \eta = 1$.

Exercise 7. *Find a nonnegative function* $\phi \in C_c^\infty(-1,1)$ *with* $\int \phi = 1$. *Can you also make* ϕ *even and strictly positive on the interior of its support?*

Exercise 8. *Let*

$$f(x) = \chi_{(-\delta,\delta)}\left(\frac{x - x_0}{\delta}\right)$$

and consider the convolution integral

$$\mu_\epsilon * f(x) = \int_{\xi\in\mathbb{R}} \mu(\xi) f(\xi - x).$$

Show the following:

(a) $\mu_\epsilon * f = f * \mu_\epsilon$.

(b) $\mu_\epsilon * f \in C_c^\infty(\mathbb{R})$.

(c) $0 \le \mu_\epsilon * f(x) \le 1$ *for all* $x \in \mathbb{R}$.

(d) *When* $\epsilon < \delta$

$$\mu_\epsilon * f(x) = \begin{cases} 0, & |x| \ge \delta + \epsilon \\ 1, & |x| \le \delta - \epsilon. \end{cases}$$

(e) *Verify (1.6).*

Exercise 9. *If* $g \in \sqsubset^1[a,b]$, *show that for each* $\tilde{x}$ *and* x *in* $[a,b]$ *one has*

$$|g(\tilde{x}) - g(x)| < \lambda|\tilde{x} - x|$$

where

$$\lambda = \sup_{\xi\in[a,b]} |g'(\xi)|.$$

What happens if $g \in \sqsubset_c^1(\mathbb{R})$*?*

Exercise 10. *Let $\bar{\mu} \in \sqsubset^1_c(\mathbb{R})$ be given by*

$$\bar{\mu}(x) = \begin{cases} -\frac{1}{2\delta}(|x| - 1 - \delta) & \text{if } 1 - \delta \leq |x| \leq 1 + \delta, \text{ and} \\ \chi_{(-1,1)} & \text{otherwise} \end{cases}$$

as indicated in Figure 1.4. Compute the explicit pointwise values of the mollification $\mu_\epsilon * \bar{\mu}$.

1.2 The Euler-Lagrange Equation

Theorem 3 (Proposition 1.5 in BGH)**.** *(The Euler-Lagrange Equation) If*

$$u \in C^2(a, b)$$

is a weak extremal for the functional

$$\mathcal{F}[u] = \int_a^b F(x, u(x), u'(x))\, dx$$

with Lagrangian $F \in C^2((a, b) \times \mathbb{R} \times \mathbb{R})$, then

$$\frac{d}{dx} F_p(x, u, u') - F_z(x, u, u') = 0 \quad \text{on} \quad (a, b). \tag{1.14}$$

Proof: We know

$$\int_a^b [F_z(x, u, u')\phi + F_p(x, u, u')\phi']\, dx = 0 \quad \text{for all } \phi \in C_c^\infty(a, b). \tag{1.15}$$

Integrating the second term by parts we get

$$\begin{aligned} \int_a^b F_p(x, u, u')\phi'\, dx &= F_p(x, u, u')\phi\Big|_a^b - \int_a^b \frac{d}{dx} F_p(x, u, u')\phi\, dx \\ &= -\int_a^b \frac{d}{dx} F_p(x, u, u')\phi\, dx. \end{aligned}$$

Replacing this expression in (1.15), we get

$$\int_a^b \left[F_z(x, u, u') - \frac{d}{dx} F_p(x, u, u') \right] \phi\, dx = 0 \quad \text{for all } \phi \in C_c^\infty(a, b).$$

Thus, the result follows from the fundamental lemma. □