Therefore,

$$\int (u + u'^2)\,dx - \int (u_0 + {u_0'}^2)\,dx = \int v'^2\,dx \geq 0$$

with equality only if $v' \equiv 0$. In this admissible class, this means $v \equiv 0$. □

1.7 Free boundary conditions

We have considered weak extremals u for which

$$\delta\mathcal{F}_u[\phi] = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

Here we consider the much stronger condition

$$\delta\mathcal{F}_u[\phi] = 0 \qquad \text{for all } \phi \in C^1[a,b]. \tag{1.26}$$

Theorem 5 (Proposition 1.12 in BGH)**.** *If $u \in C^1[a,b]$, the Lagrangian F satisfies $F \in C^1([a,b] \times \mathbb{R} \times \mathbb{R})$, and (1.26) holds, then in addition to the Euler-Lagrange equation we have also the endpoints conditions*

$$F_p(a, u(a), u'(a)) = 0 = F_p(b, u(b), u'(b)).$$

Proof: Recall that we have shown by the Lemma of DuBois-Reymond, and the extremality condition, that

$$\psi(x) = F_p(x, u, u') \quad \text{is a } C^1 \text{ function of } x \text{ on } (a,b).$$

In fact,

$$F_p(x, u, u') = F_p(a, u(a), u'(a)) + \int_a^x F_z(\xi, u(\xi), u'(\xi))\,d\xi.$$

Notice that ψ has well defined values at the endpoints a and b, and we can calculate right and left derivatives at these points:

$$\psi'(a^+) = \lim_{h \searrow 0} \frac{1}{h} \int_a^{a+h} F_z(\xi, u(\xi), u'(\xi))\,d\xi = F_z(a, u(a), u'(a)),$$

and

$$\begin{aligned}\psi'(b^-) &= \lim_{h \searrow 0} \frac{1}{(-h)} \left[\int_a^{b-h} F_z(\xi, u(\xi), u'(\xi))\,d\xi - \int_a^b F_z(\xi, u(\xi), u'(\xi))\,d\xi \right] \\ &= F_z(b, u(b), u'(b)).\end{aligned}$$

In view of these calculations, we may assume $\psi \in C^1[a,b]$ and return to the extremality condition

$$\int_a^b \left[F_p(x,u,u')\phi' + F_z(x,u,u')\phi\right]\, dx = 0.$$

Since we know extra regularity for $\psi = F_p(x,u,u')$, we can integrate by parts in the usual direction (but this time with $\phi(a)$ and/or $\phi(b)$ possibly nonzero) to get

$$F_p(x,u,u')\Big|_{x=a}^{b} + \int_a^b \left[F_z(x,u,u') - \frac{d}{dx}F_p(x,u,u')\right] dx = 0 \quad \text{for all } \phi \in C^1[a,b].$$

The integral vanishes because the integrand is (up to a sign) the operator in the Euler-Lagrange equation applied to an extremal. Therefore,

$$F_p(a,u(a),u'(a))\phi(a) = F_p(b,u(b),u'(b))\phi(b) \quad \text{for all } \phi \in C^1[a,b].$$

Taking ϕ with $\phi(a) = 1$ and $\phi(b) = 0$, we obtain

$$F_p(a,u(a),u'(a)) = 0.$$

Similarly, $F_p(b,u(b),u'(b)) = 0$. □

Exercise 21. *Extend Theorem 5 to the case where*

$$\mathbf{x} : [a,b] \to \mathbb{R}^n, \qquad \mathbf{x} \in C^1([a,b] \to \mathbb{R}^n),$$

$F \in C^1([a,b] \times \mathbb{R}^n \times \mathbb{R}^n)$, *and* $\mathbf{x}$ *is a weak extremal with free endpoints, meaning*

$$\int_a^b \left[\sum_{j=1}^n F_{p_j}(t,\mathbf{x},\mathbf{x}')\phi_j' + \sum_{j=1}^n F_{z_j}(t,\mathbf{x},\mathbf{x}')\phi_j\right] dt = 0$$

$$\textit{for all } \vec{\phi} = (\phi_1,\ldots,\phi_n) \in C^1([a,b] \to \mathbb{R}^n).$$

1.8 Technique of the first integral

The following considerations apply to C^2 (classical) extremals of **autonomous** functionals, i.e., ones with Lagrangians $F = F(z,p)$ with no explicit x dependence.