In view of these calculations, we may assume $\psi \in C^1[a,b]$ and return to the extremality condition

$$\int_a^b [F_p(x,u,u')\phi' + F_z(x,u,u')\phi]\,dx = 0.$$

Since we know extra regularity for $\psi = F_p(x,u,u')$, we can integrate by parts in the usual direction (but this time with $\phi(a)$ and/or $\phi(b)$ possibly nonzero) to get

$$F_p(x,u,u')\Big|_{x=a}^{b} + \int_a^b \left[F_z(x,u,u') - \frac{d}{dx}F_p(x,u,u')\right] dx = 0 \quad \text{for all } \phi \in C^1[a,b].$$

The integral vanishes because the integrand is (up to a sign) the operator in the Euler-Lagrange equation applied to an extremal. Therefore,

$$F_p(a,u(a),u'(a))\phi(a) = F_p(b,u(b),u'(b))\phi(b) \quad \text{for all } \phi \in C^1[a,b].$$

Taking ϕ with $\phi(a) = 1$ and $\phi(b) = 0$, we obtain

$$F_p(a,u(a),u'(a)) = 0.$$

Similarly, $F_p(b,u(b),u'(b)) = 0$. $\square$

Exercise 21. *Extend Theorem 5 to the case where*

$$\mathbf{x} : [a,b] \to \mathbb{R}^n, \qquad \mathbf{x} \in C^1([a,b] \to \mathbb{R}^n),$$

$F \in C^1([a,b] \times \mathbb{R}^n \times \mathbb{R}^n)$, *and* $\mathbf{x}$ *is a weak extremal with free endpoints, meaning*

$$\int_a^b \left[\sum_{j=1}^n F_{p_j}(t,\mathbf{x},\mathbf{x}')\phi_j' + \sum_{j=1}^n F_{z_j}(t,\mathbf{x},\mathbf{x}')\phi_j\right] dt = 0$$
$$\textit{for all } \vec{\phi} = (\phi_1,\ldots,\phi_n) \in C^1([a,b] \to \mathbb{R}^n).$$

1.8 Technique of the first integral

The following considerations apply to C^2 (classical) extremals of **autonomous** functionals, i.e., ones with Lagrangians $F = F(z,p)$ with no explicit x dependence.

Theorem 6 (first integral)**.** *If $u \in C^2[a,b]$ is an extremal for*

$$\mathcal{F}[u] = \int_a^b F(u,u')\,dx$$

with $F \in C^1(\mathbb{R}\times\mathbb{R})$, then

$$u'F_p(u,u') - F(u,u') = c \quad \textit{(constant).} \tag{1.27}$$

Note: *This is a first order ODE for u called the* **first integral equation** *or* **Erdmann's equation**. *For future reference, let us call the operator on the left $Eu = u'F_p(u,u') - F(u,u')$ or, more generally in the case of a non-autonomous Lagrangian,*

$$Eu = u'F_p(x,u,u') + F(x,u,u')$$

the **Erdmann operator**.

Proof: We begin by recalling the Euler-Lagrange equation satisfied by u in this case:

$$\frac{d}{dx}F_p(u,u') = F_z(u,u').$$

Using this, the derivative of the Erdmann operator may be simplified as follows:

$$\begin{aligned}\frac{d}{dx}\left[u'F_p(u,u') - F(u,u')\right] &= u''F_p(u,u') + u'\frac{d}{dx}F_p(u,u') - F_z(u,u')u' - F_p(u,u')u'' \\ &= u'F_z(u,u') - F_z(u,u')u' \\ &= 0. \qquad \square\end{aligned}$$

It is pointed out in BGH that one must be careful using the technique of the first integral:

> *Every classical extremal for an autonomous Lagrangian is a solution of the first integral equation, but* **the converse does not always hold**, *i.e., there can be solutions of the first integral equation which are not solutions of the Euler-Lagrange equation.*

The discussion in BGH suggests the following example which shows this cautionary situation may arise for a rather standard and well-behaved variational problem.

Example 4 (H^1 Poisson integral)**.** *Consider the functional*

$$\mathcal{F}[u] = \int (u'^2 + u^2 - u)\, dx$$

which is obtained from a Poisson integral by adding the square of an L^2 norm; adding the square of an L^2 norm to Dirichlet energy in the context of weak derivatives gives the square of a norm called the H^1 norm. The Euler-Lagrange equation for this functional is

$$2u'' = 2u - 1,$$

and the first integral equation is

$$u'^2 - u^2 + u = 0.$$

Notice that the constant function $u \equiv 1$ is a solution of the first integral equation but is not a solution of the Euler-Lagrange equation.

conserved quantities

Setting aside regularity issues and the subtlties associated with examples like that just above, a second order autonomous ODE, like the Euler-Lagrange equation in the autonomous case, is equivalent to a first order system for u and u', and the associated phase plane is often of interest. In this setting, the quantity

$$\Psi(u, u') = Eu = u' Fp(u, u') - F(u, u')$$

is called the **conserved quantity**.

Example 5. *The action integral*

$$A[\mathbf{x}] = \int_a^b \left[\frac{1}{2} m |\mathbf{x}'|^2 - \Phi(\mathbf{x})\right] dt$$

associated with the potential Φ gives rise to the conserved quantity

$$\begin{aligned}\Psi(\mathbf{x}, \mathbf{x}') &= m\mathbf{x}' \cdot \mathbf{x}' - \left[\frac{m}{2}|\mathbf{x}'|^2 - \Phi(\mathbf{x})\right] \\ &= \frac{m}{2}|\mathbf{x}'|^2 + \Phi(\mathbf{x})\end{aligned}$$

which is the instantaneous total energy of the system. The fact that the total energy is conserved in mechanical systems of this sort proves to be an extremely powerful tool in classical Newtonian mechanics.