1.9 C^2 inner variations

So far, we have restricted attention to **linear variations**. These are variations of the form

$$v(x; \epsilon) = u(x) + \epsilon\phi(x)$$

where ϕ is in some **linear perturbation class** $\mathcal{P}$, for example, $\mathcal{P} = C_c^\infty(a, b)$ or $\mathcal{P} = C^1[a, b]$. These variations have the advantages that the dependence on ϵ is simple and explicit, and there are well-understood useful linear perturbation classes available. Sometimes, especially in the context of nonlinear admissibility conditions, it is (more or less) necessary to consider more general variations. I say "more or less" for two reasons. First, if one restricts attention to the theory of the first variation, then consideration of more general variations does not (for the most part) yield any essentially new or different information. This point can be made a little more precise. A **general variation** of $u \in \mathcal{A}$ is a function $v : (a, b) \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ where $\epsilon_0 > 0$ and

$$v(x; 0) = u(x), \tag{1.28}$$

the condition (1.28) being the essential defining one. In this generality, the question of how v depends on ϵ can be very subtle, and the regularity of v in both variables x and ϵ requires careful attention. On the other hand, one has much more flexibility, in general, when it comes to admissibility conditions. Returning to the point under consideration, the **perturbation** associated with a general variation v is usually defined to be[5]

$$\varphi(x) = \frac{\partial v}{\partial \epsilon}\Big|_{\epsilon=0} = \frac{\partial v}{\partial \epsilon}(x; 0), \tag{1.29}$$

and it can be shown, for example, under rather general hypotheses that the first variation with respect to general variations, defined by

$$\delta\mathcal{F}_u = \frac{d}{d\epsilon}\mathcal{F}[v(x; \epsilon)]\Big|_{\epsilon=0}$$

depends only on the perturbation φ and is given by the formula

$$\delta\mathcal{F}_u[\varphi] = \int_a^b [F_z(x, u, u')\varphi + F_p(x, u, u')\varphi'] \, dx.$$

[5]More generally, the partial derivative $\partial v/\partial\epsilon$ defined at a general $(x; \epsilon)$ is sometimes called the **variation vector**.

It will be noticed at once that the first variation formula for a general variation is essentially identical to the one appearing in the condition (1.1) of Proposition 1 for a linear variation; the only difference is that the linear perturbation ϕ of (1.1) is replaced by the general perturbation φ of (1.29).

The second reason general variations may be of somewhat secondary interest is that they are rarely used in applied mathematics, i.e., by applied mathematicians. Even when the **second variation** is considered to determine (or define) stability of a physical system and the use of general variations can make a difference, applied mathematicians tend to prefer the use of linear variations and the associated condition of **linear stability** as opposed to the more natural (though more complicated) notion of **nonlinear stability** associated with general variations. It may be noted, in this connection, that the term **stability** is used for both concepts in the literature, but linear stability is what most authors (and especially applied mathematicians) usually have in mind.

We now proceed to consider an interesting special class of general variations which gives some insight into the origin of the Erdmann operator and the first integral equation; the discussion also leads to a generalization called **Noether's equation**.

A C^2 **parameter variation** of the interval $[a, b]$ is a function

$$\xi \in C^2(\,[a, b] \times (-\epsilon_0, \epsilon_0)\,)$$

satisfying

$$\frac{\partial \xi}{\partial x} > 0, \quad \xi(a; \epsilon) \equiv a, \quad \xi(b; \epsilon) \equiv b, \qquad \text{and} \tag{1.30}$$

$$\xi(x; 0) = \mathrm{id}_{[a,b]} \equiv x. \tag{1.31}$$

The partial derivative

$$\psi = \frac{\partial \xi}{\partial \epsilon}\Big|_{\epsilon=0} = \frac{\partial \xi}{\partial \epsilon}(x; 0) \tag{1.32}$$

is called the **parameter perturbation** or inner perturbation. Given a C^2 parameter variation and and extremal $u \in C^1[a, b]$, and **inner variation** (or C^2 inner variation) is a general variation of the form

$$v(x; \epsilon) = u(\xi(x; \epsilon)). \tag{1.33}$$

When considered as a general variation, an inner variation has the peculiarity that the associated perturbation depends on the varied function:

$$\varphi(x) = u'(x)\frac{\partial \xi}{\partial \epsilon}(x; 0) = u'(x)\psi(x).$$

For this reason, the first variation functional associated with an inner variation is given a different notation by the authors of BGH:

$$\partial\mathcal{F}_u[\psi] = \frac{d}{d\epsilon}\int_a^b F\left(x, v, \frac{\partial v}{\partial x}\right)\, dx\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\int_a^b F\left(x, u(\xi), u'(\xi)\frac{\partial \xi}{\partial x}\right)\, dx\Big|_{\epsilon=0}.$$

They also give this expression a special name: **the first inner variation of $\mathcal{F}$ in the direction** ψ. If u has additional regularity, say $u \in C^2[a,b]$, then we can write

$$\partial\mathcal{F}_u[\psi] = \delta\mathcal{F}_u[u'\psi].$$

We will follow up on this situation after we establish the main results on inner variations from BGH.

1.9.1 Generalized Erdmann's equation

We first establish a formula for the first inner variation.

Theorem 7 (Proposition 1.14 in BGH)**.** *(First inner variation formula) If $u \in C^1[a,b]$ and $v = v(x;\epsilon)$ is a C^2 inner variation of u with parameter perturbation ψ, then*

$$\partial\mathcal{F}_u[\psi] = \int_a^b [(Eu)\psi' - F_x(x,u,u')\psi]\, dx \tag{1.34}$$

where

$$\mathcal{F}[u] = \int_a^b F(x,u,u')\, dx$$

is the usual integral functional with C^1 Lagrangian and

$$Eu = u'F_p(x,u,u') - F(x,u,u')$$

is the Erdmann operator appearing in the first integral equation.

Proof: For each fixed ϵ, the parameter transformation $x \mapsto \xi(x;\epsilon)$ has an inverse, which we denote $\xi^{-1} = \xi^{-1}(\eta;\epsilon)$. Differentiating the relation

$$\xi^{-1}(\xi(x)) = x \tag{1.35}$$

with respect to x we find

$$\frac{\partial \xi^{-1}}{\partial \eta}(\xi(x))\frac{\partial \xi}{\partial x} = 1 \qquad \text{or} \qquad \frac{\partial \xi}{\partial x} = \frac{1}{\frac{\partial \xi^{-1}}{\partial \eta}(\xi(x))}.$$

Starting with the definition of the first inner variation, we change variables using $\eta = \xi(x;\epsilon)$ so that

$$d\eta = \frac{\partial \xi}{\partial x}\, dx \qquad \text{and} \qquad dx = \frac{\partial \xi^{-1}}{\partial \eta}(\eta)\, d\eta.$$

Thus,

$$\begin{aligned}
\partial \mathcal{F}_u[\psi] &= \frac{d}{d\epsilon} \int_a^b F\left(x, u(\xi), u'(\xi)\frac{\partial \xi}{\partial x}\right) dx\Big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon} \int_a^b F\left(\xi^{-1}, u(\eta), u'(\eta)\frac{1}{\frac{\partial \xi^{-1}}{\partial \eta}}\right) \frac{\partial \xi^{-1}}{\partial \eta}\, d\eta\Big|_{\epsilon=0}.
\end{aligned}$$

The dependence on ϵ in the integrand is now isolated in the appearances of ξ^{-1}, and we have

$$\begin{aligned}
\partial \mathcal{F}_u[\psi] = \int_a^b \Bigg\{ \Bigg[& F_x\left(\xi^{-1}, u(\eta), u'(\eta)\frac{1}{\frac{\partial \xi^{-1}}{\partial \eta}}\right) \frac{\partial \xi^{-1}}{\partial \epsilon} \\
& - F_p\left(\xi^{-1}, u(\eta), u'(\eta)\frac{1}{\frac{\partial \xi^{-1}}{\partial \eta}}\right) \frac{u'(\eta)}{\left(\frac{\partial \xi^{-1}}{\partial \eta}\right)^2} \frac{\partial^2 \xi^{-1}}{\partial \epsilon \partial \eta} \Bigg] \frac{\partial \xi^{-1}}{\partial \eta} \\
& + F\left(\xi^{-1}, u(\eta), u'(\eta)\frac{1}{\frac{\partial \xi^{-1}}{\partial \eta}}\right) \frac{\partial^2 \xi^{-1}}{\partial \epsilon \partial \eta} \Bigg\}\, d\eta\Big|_{\epsilon=0}
\end{aligned}$$

Since $\xi(x;0) = \mathrm{id}_{[a,b]} \equiv x$, we have also

$$\xi^{-1}(\eta;0) = \mathrm{id}_{[a,b]} \equiv \eta \qquad \text{and} \qquad \frac{\partial \xi^{-1}}{\partial \eta}(\eta;0) \equiv 1.$$

Furthermore, we can differentiate (1.35) with respect to ϵ to obtain

$$\frac{\partial \xi^{-1}}{\partial \eta}\frac{\partial \xi}{\partial \epsilon} + \frac{\partial \xi^{-1}}{\partial \epsilon} = 0 \qquad \text{or} \qquad \frac{\partial \xi^{-1}}{\partial \epsilon} = -\frac{\partial \xi^{-1}}{\partial \eta}\frac{\partial \xi}{\partial \epsilon}.$$

Evaluating at $\epsilon = 0$, this means

$$\frac{\partial \xi^{-1}}{\partial \epsilon} = -\psi(\eta).$$

Similarly,

$$\frac{\partial^2 \xi^{-1}}{\partial \eta \partial \epsilon}(\eta; 0) = -\psi'(\eta).$$

Making these substitutions in our expression for $\partial \mathcal{F}_u[\psi]$, we have

$$\begin{aligned}\partial \mathcal{F}_u[\psi] = \int_a^b \{[&F_x(\eta, u(\eta), u'(\eta))(-\psi) - F_p(\eta, u(\eta), u'(\eta))\, u'(\eta)(-\psi')] \\ &+ F(\eta, u(\eta), u'(\eta))(-\psi')\}\, d\eta.\end{aligned}$$

Renaming the variable of integration η to x and rearranging the terms, we see this is (1.34). □

Theorem 8 (Proposition 1.16 in BGH[6]). *If $u \in C^1[a, b]$ and*

$$\partial \mathcal{F}_u[\psi] = 0 \quad \textit{for all } C^2 \textit{ inner variations,}$$

then there is some constant c such that

$$Eu = u' F_p(x, u, u') - F(x, u, u') = c - \int_a^x F_x(\eta, u(\eta), u'(\eta))\, d\eta$$

and

$$\frac{d}{dx}\left[u' F_p(x, u, u') - F(x, u, u')\right] = -F_x(x, u, u').$$

The first equation, involving Erdmann's first order operator, generalizes the theorem of the first integral. The second equation is called Noether's equation. As with the first integral equation, this equation, presumably, should not be considered equivalent to the Euler-Lagrange equation, but rather a second order equation with possibly (many) more solutions than the Euler-Lagrange equation.

Proof: For any $\psi \in C_c^\infty$ the parameter variation

$$\xi(x; \epsilon) = x + \epsilon\psi$$

is well-defined for all ϵ small enough. That is, $u(\xi(x; \epsilon))$ is an inner variation with inner perturbation

$$\frac{\partial \xi}{\partial \epsilon}(x; 0) = \psi(x).$$

[6]There seems to be a sign error in the statement of this result in BGH; if the function Φ used in BGH is replaced with what we have called the Erdmann operator, the statement seems to be correct.

According to (1.34) we have shown

$$\int_a^b [(Eu)\psi' - F_x(x,u,u')\psi]\,dx = 0 \qquad \text{for all } \psi \in C_c^\infty(a,b).$$

Integrating by parts we get

$$\int_a^b [Eu + g]\psi'\,dx = 0 \qquad \text{for all } \psi \in C_c^\infty(a,b)$$

where

$$g(x) = \int_a^x F_x(\eta, u(\eta), u'(\eta))\,d\eta.$$

Notice that $g \in C^1[a,b]$. It now follows from the Lemma of DuBois-Reymond that there is some constant c for which

$$Eu = c - g(x).$$

This establishes the first assertion of the theorem. Because the right side $c - g(x)$ is a C^1 function, the left side is differentiable too, and Noether's equation follows from differentiation. $\square$

1.9.2 Inner variations with higher regularity

We used a change of variables in the calculation of the first inner variation formula which we then used to derive the generalized first integral relations. If we have additional regularity, we should be able to make this calculation without the change of variables. Let us verify that this is the case. If $u \in C^2[a,b]$ and $\xi \in C^2([a,b]\times(-\epsilon_0,\epsilon_0))$ is a parameter variation with associated inner variation $v = u(\xi(x;\epsilon))$, then the formula for the first inner variation becomes

$$\begin{aligned}
\partial\mathcal{F}_u[\psi] &= \frac{d}{d\epsilon}\int_a^b F\left(x, v, \frac{\partial v}{\partial x}\right)\,dx\Big|_{\epsilon=0} \\
&= \int_a^b \left[(\, u'F_z(x,u,u') + u''F_p(x,u,u')\,)\,\frac{\partial \xi}{\partial \epsilon} + u'F_p(x,u,u')\frac{\partial^2 \xi}{\partial x \partial \epsilon}\right]\,dx \\
&= \int_a^b [(\, u'F_z(x,u,u') + u''F_p(x,u,u')\,)\,\psi + u'F_p(x,u,u')\psi']\,dx.
\end{aligned}$$

Notice that

$$\frac{d}{dx}F(x,u,u') = F_x(x,u,u') + u'F_z(x,u,u') + u''F_p(x,u,u').$$

By adding and subtracting the integral

$$\int_a^b F_x(x,u,u')\psi\,dx,$$

integrating by parts, and applying the lemma of DuBois-Reymond, we arrive at the same conclusions of Theorem 8 under the additional regularity assumptions. See Exercise 22

Exercise 22. *Carry out the details of the proof of Theorem 8 suggested above under the additional regularity assumption $u \in C^2[a,b]$.*

Exercise 23. *An inner variation is said to be C_c^∞ if there is some interval $I \subset\subset (a,b)$ such that*

$$\xi\Big|_{[a,b]\setminus\bar{I}} \equiv x.$$

Give condituions on the parameter variation ξ and the function u under which there is some $\varphi \in C_c^\infty(a,b)$ such that the inner variation $v = u(\xi(x;\epsilon))$ satisfies the formula

$$\partial\mathcal{F}_u\left[\frac{\partial\xi}{\partial\epsilon}(x;0)\right] = \delta\mathcal{F}_u[\varphi].$$

1.10 Variational constraints; Lagrange multipliers

It is quite common to encounter a variational problem with an **integral constraint** of the form

$$\mathcal{G}[u] = \int_a^b G(x,u,u')\,dx = g_0 \qquad \text{(constant)}.$$

For example, rather than looking for the shortest graph connecting $(a,0)$ to $(b,0)$ in the plane, which is easily seen to be given by the graph of $u(x) \equiv 0$, one may look for the shortest graph connecting these two points among graphs enclosing (along with the segment along the axis between the two