Notice that

$$\frac{d}{dx}F(x,u,u') = F_x(x,u,u') + u'F_z(x,u,u') + u''F_p(x,u,u').$$

By adding and subtracting the integral

$$\int_a^b F_x(x,u,u')\psi\,dx,$$

integrating by parts, and applying the lemma of DuBois-Reymond, we arrive at the same conclusions of Theorem 8 under the additional regularity assumptions. See Exercise 22

Exercise 22. *Carry out the details of the proof of Theorem 8 suggested above under the additional regularity assumption $u \in C^2[a,b]$.*

Exercise 23. *An inner variation is said to be C_c^∞ if there is some interval $I \subset\subset (a,b)$ such that*

$$\xi\big|_{[a,b]\setminus\bar{I}} \equiv x.$$

Give condituions on the parameter variation ξ and the function u under which there is some $\varphi \in C_c^\infty(a,b)$ such that the inner variation $v = u(\xi(x;\epsilon))$ satisfies the formula

$$\partial\mathcal{F}_u\left[\frac{\partial\xi}{\partial\epsilon}(x;0)\right] = \delta\mathcal{F}_u[\varphi].$$

1.10 Variational constraints; Lagrange multipliers

It is quite common to encounter a variational problem with an **integral constraint** of the form

$$\mathcal{G}[u] = \int_a^b G(x,u,u')\,dx = g_0 \qquad \text{(constant)}.$$

For example, rather than looking for the shortest graph connecting $(a,0)$ to $(b,0)$ in the plane, which is easily seen to be given by the graph of $u(x) \equiv 0$, one may look for the shortest graph connecting these two points among graphs enclosing (along with the segment along the axis between the two points) a prescribed area A. It is natural to include such a constraint in the admissible class and write, for example,

$$\mathcal{A} = \left\{ u \in C^1[a,b] : u(a) = 0 = u(b),\ \int_a^b u(x)\,dx = A \right\}.$$

We will now give an alternative which allows us to use the methods developed above for unconstrained problems. The main result is the following:

Theorem 9 (Proposition 1.17 in BGH)**.** *Let*

$$\mathcal{A} = \left\{ u \in C^0[a,b] \cap C^1(a,b) : u(a) = u_a,\ u(b) = u_b,\ \text{and } \mathcal{G}[u] = g_0 \right\}$$

where

$$\mathcal{G}[u] = \int_a^b G(x,u,u')\,dx \qquad \text{with } G \in C^1([a,b] \times \mathbb{R} \times \mathbb{R}).$$

Assume $u_0 \in \mathcal{A}$ *satisfies*

$$\mathcal{F}[u_0] \le \mathcal{F}[u] \qquad \text{for all } u \in \mathcal{A}$$

where

$$\mathcal{F}[u] = \int_a^b F(x,u,u')\,dx \qquad \text{with } F \in C^1([a,b] \times \mathbb{R} \times \mathbb{R}).$$

Then either $\delta\mathcal{G}_{u_0}[\phi] \equiv 0$ *for all* $\phi \in C_c^\infty(a,b)$*, or there is some* $\lambda \in \mathbb{R}$ *such that*

$$\delta\mathcal{F}_{u_0}[\phi] + \lambda\,\delta\mathcal{G}_{u_0}[\phi] = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

In the latter case, if either

1. $u_0 \in C^1[a,b]$*, or*

2. $u_0 \in C^2(a,b)$ *and* $F, G \in C^2((a,b) \times \mathbb{R} \times \mathbb{R})$*,*

then u_0 *is a solution of the Euler-Lagrange equation for the functional* $\mathcal{F} + \lambda\,\mathcal{G}$*. Roughly speaking, this says the constrained problem for* $\mathcal{F}$ *is equivalent to the unconstrained problem for* $\mathcal{F} + \lambda\,\mathcal{G}$*.*

Proof: For the proof, denote u_0 by u; we need make no comparison requiring a distinction between u_0 and u as stated in the theorem. If $\delta\mathcal{G}_u \not\equiv 0$, then there is some $\psi \in C_c^\infty(a,b)$ such that

$$\delta\mathcal{G}_u[\psi] = 1.$$

For ψ (fixed) and $\phi \in C_c^\infty(a,b)$ arbitrary (but also temporarily fixed), consider the functions $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(\epsilon,\delta) = \mathcal{F}[u + \epsilon\phi + \delta\psi] \qquad \text{and} \qquad g(\epsilon,\delta) = \mathcal{G}[u + \epsilon\phi + \delta\psi].$$

Notice that $g(0,0) = \mathcal{G}[u] = g_0$ and

$$\frac{\partial g}{\partial \delta}(0,0) = \delta\mathcal{G}_u[\psi] = 1.$$

By the implicit function theorem the equation $g(\epsilon,\delta) = g_0$ determines δ as a C^1 function of ϵ locally near $(0,0)$. To be more precise, there is some $\epsilon_0 > 0$ and a unique C^1 function $h : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ such that

$$g(\epsilon, h(\epsilon)) = g_0 \qquad \text{for } |\epsilon| < \epsilon_0. \tag{1.36}$$

This means we have a one-parameter family of admissible functions

$$u + \epsilon\phi + h(\epsilon)\psi \qquad \text{for } |\epsilon| < \epsilon_0,$$

and

$$f(0,0) = \mathcal{F}[u] \le \mathcal{F}[u + \epsilon\phi + h(\epsilon)\psi] = f(\epsilon, h(\epsilon)).$$

Consequently,

$$\frac{d}{d\epsilon} f(\epsilon, h(\epsilon))\Big|_{\epsilon=0} = \delta\mathcal{F}_u[\phi] + h'(0)\delta\mathcal{F}_u[\psi] = 0. \tag{1.37}$$

On the other hand, differentiating (1.36) we have

$$\delta\mathcal{G}_u[\phi] + \delta\mathcal{G}_u[\psi]h'(0) = 0, \qquad \text{so} \qquad h'(0) = -\delta\mathcal{G}_u[\phi].$$

Thus, (1.37) implies

$$\delta\mathcal{F}_u[\phi] - \delta\mathcal{F}_u[\psi]\delta\mathcal{G}_u[\phi] = 0 \qquad \text{for all } \phi \in C_c^\infty(a,b).$$

Taking $\lambda = -\delta\mathcal{F}_u[\psi]$, we have established the first assertion of the theorem.

The Lagrangians may now be combined into a single Lagrangian

$$F(x, u, u') + \lambda G(x, u, u')$$

to which either Theorem 4 may be applied since

$$\begin{aligned}\mathcal{A}_1 = \{u \in C^1[a, b] : u(a) = u_a,\ u(b) = u_b\}\\ \subset \mathcal{A}_0 = \{u \in C^0[a, b] \cap C^1(a, b) : u(a) = u_a,\ u(b) = u_b\},\end{aligned}$$

or alternatively, if we have the extra regularity with $u \in C^2(a, b)$ and $F, G \in C^2((a, b) \times \mathbb{R} \times \mathbb{R})$, then Theorem 3 applies. $\square$

There is a result which should be included here. It gives some substance to the claim above that consideration of the unconstrained functional $\mathcal{F} + \lambda\mathcal{G}$ is fundamentally related to the constrained minimization of $\mathcal{F}$ with respect to $\mathcal{G}$. It also removes many of the complications of Theorem 9.

Theorem 10. *If u_0 minimizes the functional $\mathcal{F} + \lambda\mathcal{G}$ without any constraint and it happens to be the case that the minimizer satisfies a particular constraint $\mathcal{G}[u_0] = g_0$, then u is also a minimizer of $\mathcal{F}$ with respect to the constraint $\mathcal{G}[u_0] = g_0$.*

Proof: If u satisfies $\mathcal{G}[u] = g_0$, then since we know

$$\mathcal{F}[u_0] + \lambda\mathcal{G}[u_0] \le \mathcal{F}[u] + \lambda\mathcal{G}[u]$$

we can cancel the terms $\lambda\mathcal{G}[u_0] = \lambda g_0 = \lambda\mathcal{G}[u]$, and

$$\mathcal{F}[u_0] \le \mathcal{F}[u]. \qquad \square$$

Example 6 (Example 6 on page 24 of BGH)**.** *The variational problem associated with the potential equation (discussed in Example 1.5.4 of § 1.5) is often considered subject to a constraint on the L^2 norm:*

$$\mathcal{G}[u] = \frac{1}{2}\int_a^b [u(x)]^2\, dx = g_0.$$

The functional here is

$$\mathcal{F}[u] = \frac{1}{2}\int_a^b \left[u'^2 + c(x)u^2\right]\, dx.$$

The augmented functional is

$$\mathcal{F}[u] - \lambda \mathcal{G} = \frac{1}{2} \int_a^b \left[u'^2 + c(x) u^2 - \lambda u^2 \right] \, dx,$$

and the associated Euler-Lagrange equation is

$$-u'' + c(x) u = \lambda u. \tag{1.38}$$

If considered on the admissible class with fixed endpoints,

$$\mathcal{A} = \left\{ u \in C^1[a,b] : u(a) = 0 = u(b), \ \frac{1}{2} \int_a^b u^2 \, dx = g_0 \right\},$$

we are led to a two point boundary value problem for the equation in (1.38) in which the constant λ is also an unknown. This kind of problem is called a **Sturm-Liouville problem***, and the operator*

$$Lu = -u'' + c(x) u$$

is called a **Sturm-Liouville operator***. Notice the Euler-Lagrange equation prescribes that λ is an eigenvalue for the associated Sturm-Liouville operator.*

Exercise 24. *Take the special case $c(x) \equiv 1$ with $u_a = 0 = u_b$. What does the theory of ODEs say about the original minimization problem for $\mathcal{F}$ over $\mathcal{A}$? Can your conclusions be generalized to other potentials $c = c(x)$ and other boundary values?*