Chapter 2

The second variation; stability

Here we assume the Lagrangian $F \in C^2([a,b] \times \mathbb{R} \times \mathbb{R})$ but still we generally require no more of an extremal u then $u \in C^1[a,b]$; sometimes less. If, as is natural to assume, u is a local minimizer and the admissible class $\mathcal{A}$ admits the comparison

$$\mathcal{F}[u] \leq \mathcal{F}[u+\phi] \quad \text{for all } \phi \in C_c^\infty(a,b) \tag{2.1}$$

with $|\phi|_{C^1}$ small enough, then

$$\delta^2 \mathcal{F}_u[\phi] := \frac{d^2}{d\epsilon^2} \mathcal{F}[u+\epsilon\phi]\Big|_{\epsilon=0} \geq 0 \quad \text{for all } \phi \in C_c^\infty(a,b). \tag{2.2}$$

The functional $\delta^2 \mathcal{F}_u : C_c^\infty(a,b) \to \mathbb{R}$ is called the **second variation** of $\mathcal{F}$ at u in the direction ϕ. Computing, we find

$$\begin{aligned} \delta^2 \mathcal{F}_u[\phi] &= \frac{d}{d\epsilon} \int_a^b [F_z(x, u+\epsilon\phi, u'+\epsilon\phi')\phi + F_p((x, u+\epsilon\phi, u'+\epsilon\phi')\phi'] \, dx\Big|_{\epsilon=0} \\ &= \int_a^b [F_{zz}(x,u,u')\phi^2 + 2F_{zp}(x,u,u')\phi\phi' + F_{pp}(x,u,u')\phi'^2] \, dx. \end{aligned}$$

Definition 4 (stability)**.** *An extremal $u \in C^1(a,b)$ is said to be* **stable** *if*

$$\delta^2 \mathcal{F}[\phi] > 0 \quad \textit{for all } \phi \in C_c^\infty(a,b) \backslash \{0\}.$$

The extremal u is said to **unstable** *otherwise.*[1]

[1]Here we have given the simplest case, which applies to a problem with fixed endpoints and no constraint. The definition requires modification if one wishes to define stability for these more general problems. Some authors call our definition of stability "strict stability" and allow an extremal with $\delta^2 \mathcal{F}_u \geq 0$ to be called "stable" or "neutrally stable" if it is known that equality is attained.

While $\delta\mathcal{F}_u$ was a linear functional, we see $\delta^2\mathcal{F}_u$ is a quadratic form. As a consequence, there is an associated bilinear form which naturally extends to the Hilbert space $H^1(a,b)$:

$$B(\phi,\psi) = \int_{(a,b)} [F_{zz}(x,u,u')\phi\psi + F_{zp}(x,u,u')(\phi\psi' + \phi'\psi) + F_{pp}(x,u,u')\phi'\psi'].$$

We will discuss this bilinear form and its relation to stability later, but for now, let us derive a pointwise necessary condition satisfied by a minimizer (and the Lagrangian admitting it).

Theorem 11. *If u is an extremal satisfying (2.1), then*

$$F_{pp}(x,u,u') \geq 0.$$

This says, roughly, that F is convex in p along the extremal; it is called the **Legendre condition**. *Notice that this is forcing F to have some kind of convexity. One can never find a minimizer if $F_{pp} < 0$.*

Proof: Here again, the authors of BGH use that $C_c^\infty(a,b)$ is dense in some other space in order to get the condition (2.2) to hold for a nonsmooth function. In this case, they consider the "tent" function $v \in \sqsubset^1[a,b]$ given by

$$v(x) = \begin{cases} 0, & |x-x_0| \geq \delta \\ 1 - |x-x_0|/\delta, & |x-x_0| \leq \delta \end{cases}$$

where $x_0 \in (a,b)$ is fixed and δ is small enough so that the interval $[x_0 - \delta, x_0+\delta] \subset (a,b)$. It is clear that the formula for $\delta^2\mathcal{F}_u[v]$ makes sense and gives a finite value. Let us assume, for the moment, the value is always nonnegative and justify that assumption later.

Let $\epsilon > 0$ be arbitrary. The functions $F_{zz}(x,u,u')$, $F_{zp}(x,u,u')$, and $F_{pp}(x,u,u')$ are uniformly continuous on $[a,b]$. This means, in particular, that if δ is small enough then for each $x \in [x_0-\delta, x_0+\delta]$ we will have

$$\begin{aligned} |F_{zz}(x,u,u') - F_{zz}(x_0,u(x_0),u'(x_0))| &\leq \epsilon \\ |F_{zp}(x,u,u') - F_{zp}(x_0,u(x_0),u'(x_0))| &\leq \epsilon \\ |F_{pp}(x,u,u') - F_{pp}(x_0,u(x_0),u'(x_0))| &\leq \epsilon. \end{aligned}$$

In fact, we don't really need uniform continuity for this, but only continuity at x_0. Applying this to the first term

$$\int_a^b F_{zz}(x,u,u')v^2\,dx = \int_{x_0-\delta}^{x_0+\delta} F_{zz}(x,u,u')v^2\,dx$$

in $\delta^2\mathcal{F}_u[v]$, we can write

$$\left|\int_a^b F_{zz}(x,u,u')v^2\,dx\right| \leq \left|\int_a^b F_{zz}(x_0,u(x_0,u'(x_0))v^2\,dx\right| + \epsilon\int_a^b v^2\,dx$$
$$= (|F_{zz}(x_0,u(x_0,u'(x_0))|+\epsilon)\int_a^b v^2\,dx.$$

Furthermore,

$$\int_a^b v^2\,dx = 2\int_{x_0-\delta}^{x_0}\frac{(\delta+x-x_0)^2}{\delta^2}\,dx = \frac{2}{3\delta^2}\delta^3 = \frac{2\delta}{3}.$$

We conclude that

$$\lim_{\delta\searrow 0}\left|\int_a^b F_{zz}(x,u,u')v^2\,dx\right| = 0.$$

Estimating the second term similarly,

$$2\int_a^b F_{zp}(x,u,u')vv'\,dx = 2F(x_0,u(x_0),u'(x_0))\int_a^b vv'\,dx$$
$$+\,2\int_a^b [F_{zp}(x,u,u') - F_{zp}(x_0,u(x_0),u'(x_0))]vv'\,dx$$
$$= 2\int_a^b [F_{zp}(x,u,u') - F_{zp}(x_0,u(x_0),u'(x_0))]vv'\,dx$$

since

$$\int_a^b vv'\,dx = \frac{1}{\delta}^2\left[\int_{x_0-\delta}^{x_0}(x-x_0+\delta)\,dx - \int_{x_0}^{x_0+\delta}(-x+x_0+\delta)\,dx\right] = 0.$$

Therefore,

$$\left|\int_a^b F_{zp}(x,u,u')vv'\,dx\right| \leq \epsilon\int_a^b |vv'|\,dx,$$

and

$$\int_a^b |vv'|\,dx = \frac{2}{\delta^2}\int_{x_0-\delta}^{x_0}(x-x_0+\delta)\,dx = 1.$$

The middle term in $\delta^2\mathcal{F}_u[v]$ is thus estimated by

$$2\left|\int_a^b F_{zp}(x,u,u')vv'\,dx\right| \leq 2\epsilon.$$

Finally, we consider the last term:

$$\int_a^b F_{pp}(x,u,u')v'^2\,dx = \frac{1}{\delta^2}\int_{x_0-\delta}^{x_0+\delta} F_{pp}(x,u,u')\,dx.$$

We conclude that for $\delta > 0$,

$$0 \leq \delta\int_a^b F_{zz}(x,u,u')\phi^2\,dx + 2\delta\int_a^b F_{zp}(x,u,u')\phi\phi'\,dx + +2\,\frac{1}{2\delta}\int_{x_0-\delta}^{x_0^delta} F_{pp}(x,u,u')\,dx.$$

The first two terms limit as $\delta \searrow 0$ to zero, and the last term is twice the average value of $F_{pp}(x,u,u')$ over the interval $(x_0-\delta, x_0+\delta)$. The average tends to $F_{pp}(x_0, u(x_0), u'(x_0))$ as $\delta \searrow 0$. The Legendra condition follows. Notice that we didn't need the ϵ estimate for F_{pp}; in fact, we didn't need a continuity estimate for F_{zz} and F_{zp} with an arbitrary epsilon either; we only needed a finite bound. We are done except for showing

$$\int_a^b [F_{zz}(x,u,u')v^2 + 2F_{zp}(x,u,u')vv' + F_{pp}(x,u,u')v'^2]\,dx \geq 0. \qquad \square \quad (2.3)$$

We mentioned the C^1 norm on $C_c^\infty(a,b)$ above, as well as the space $H^1(a,b)$. The function v used above and the estimates needed to show (2.3) are closely related to this norm and this space, so we might as well give a little broader discussion in the course of establishing (2.3).