

## CHAPTER 6

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### The Binomial Method for Option Pricing

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We shall postulate that the underlying asset can be modeled by geometric Brownian motion

$$dS = \mu S dt + \sigma S dW$$

where  $\mu$  is the so-called drift rate. We have seen that if we apply Ito's lemma to the function

$$u(S) = \ln S$$

then  $u(S(t))$  satisfies the constant coefficient equation

$$du = (\mu - \sigma^2/2)dt + \sigma dW$$

which has the solution

$$u(t) - u(t_0) = (\mu - \sigma^2/2)(t - t_0) + \sigma(W(t) - W(t_0)).$$

Hence the asset price at time  $t$  is the random number

$$S(t) = S(t_0) \exp [(\mu - \sigma^2/2)(t - t_0) + \sigma(W(t) - W(t_0))].$$

For convenience we shall set  $t_0 = 0$ . Then

$$W(t) - W(0) = W(t) = \sqrt{t} \phi$$

where  $\phi \in N(0, 1)$ .

The analytic expression for  $S(t)$  shows that  $S(t) > 0$  if  $S(0) > 0$  and that  $S(t) = 0$  if  $S(0) = 0$  regardless of the value of  $W(t)$ .

The analytic formula for  $S(t)$  allows us to find the probability density function for  $S(t)$ .

We observe that

$$\begin{aligned} P(\ln S(t)/S(0) \leq y) &= P((\mu - \sigma^2/2)t + \sigma W(t) \leq y) \\ &= P\left(\phi < \frac{y - (\mu - \sigma^2/2)t}{\sigma\sqrt{t}}\right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{y - (\mu - \sigma^2/2)t}{\sigma\sqrt{t}}} e^{-z^2/2} dz. \end{aligned}$$

Hence

$$\begin{aligned} P(S(t) \leq x) &= P(\ln S(t)/S(0) \leq \ln x/S(0)) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln x/S(0) - (\mu - \sigma^2/2)t}{\sigma\sqrt{t}}} e^{-z^2/2} dz \end{aligned}$$

so that

$$pdf(x) = \frac{dP(S < x)}{dx} = \frac{1}{\sigma\sqrt{2\pi} x\sqrt{t}} e^{-[(\ln x/S(0) - (\mu - \sigma^2/2)t)/(\sigma\sqrt{t})]^2/2}$$

for  $x \in (0, \infty)$ .

Given  $S(0)$  the mean for  $S(t)$  can be computed from

$$E(S(t) | S(0)) = \int_0^{\infty} x pdf(x) dx.$$

With the change of variable

$$z = \frac{\ln(x/S(0)) - (\mu - \sigma^2/2)t}{\sigma\sqrt{t}}$$

the integral becomes

$$E(S(t) | S(0)) = \frac{1}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-z^2/2} x \sigma\sqrt{t} dz$$

where

$$x = S(0)e^{\sigma\sqrt{t}z} e^{(\mu - \sigma/2)t}$$

It is a common trick to complete the square and write

$$e^{-x^2+bx} = e^{(b/2)^2-u^2}$$

with  $u = x - b/2$ , in order to transform the above integral into a standard probability density integral. With  $u = (z + \sigma\sqrt{t})$  we find that

$$E(S(t) | S(0)) = S(0)e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} du$$

so that

$$E(S(t) | S(0)) = S(0)e^{\mu t}.$$

Thus the drift  $\mu$  reflects our view how the mean of the asset will evolve with time.  $\mu$  is not observable and will differ from one asset to the next.

An analogous manipulation applied to

$$E(S^2(t) | S(0)) = \int_0^\infty x^2 pdf(x) dx$$

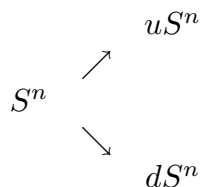
will yield the result

$$E(S^2(t) | S(t)) = S^2(0)e^{2\mu t + \sigma^2 t}$$

so that

$$\text{var}(S(t) | S(0)) = S^2(0)e^{2\mu t}(e^{\sigma^2 t} - 1).$$

Let us now simulate this random motion of  $S(t)$  with a time-discrete motion where the asset value  $S^n$  at time  $t = t_n$  can rise at time  $t_n + \Delta t$  to  $uS^n$  with probability  $p$  or fall to  $dS^n$  with probability  $(1 - p)$ . Schematically, we have



We impose on the rise and fall of  $S$  the requirement that over the time interval  $\Delta t$  the expected values of  $uS$  and  $dS$  and their variances are identical to those of the continuous motion starting with  $S^n$  at  $t = t^n$ . This will be the case if

$$up + d(1 - p) = e^{\mu\Delta t}$$

$$u^2p + d^2(1 - p) = e^{(2\mu + \sigma^2)\Delta t}.$$

These are two equations in the three parameters  $u$ ,  $d$ ,  $p$ . They become uniquely defined with the additional assumption that a downward move will cancel a preceding upward move, i.e.

$$ud = 1.$$

A little bit of algebra will now yield values for  $u$  and  $d$ . We find from

$$p = \frac{e^{\mu\Delta t} - d}{u - d} \quad \text{and} \quad p = \frac{e^{(2\mu + \sigma^2)\Delta t} - d^2}{(u + d)(u - d)}$$

and

$$ud = 1$$

that

$$u = A + \sqrt{A^2 - 1}, \quad d = A - \sqrt{A^2 - 1}$$

with

$$A = \frac{e^{-\mu\Delta t} + e^{(\mu+\sigma^2)\Delta t}}{2}.$$

We note that in this model

$$(6.1) \quad p = \frac{e^{\mu\Delta t} - d}{u - d}$$

is supposed to represent the probability of an upward jump. Hence we require  $0 < p < 1$ . This will be the case for all  $\Delta t$  because

$$(6.2) \quad p(\Delta t) = \frac{e^{\mu\Delta t} - A + \sqrt{A^2 - 1}}{2\sqrt{A^2 - 1}} = \frac{1}{2} - \frac{e^{\mu\Delta t} - A}{2\sqrt{A^2 - 1}}$$

and l'Hospital's rule imply that

$$\lim_{\Delta t \rightarrow 0} p(\Delta t) = \frac{1}{2}.$$

Moreover  $p(\Delta t) = 0$  for some positive  $\Delta t$  would imply that

$$e^{\mu\Delta t} = d = A - \sqrt{A^2 - 1}$$

or

$$e^{\mu\Delta t} (A + \sqrt{A^2 - 1}) = 1$$

which is impossible since  $A > 1$  for  $\Delta t > 0$ . Similarly, if  $p(\Delta t) = 1$  then

$$e^{\mu\Delta t} = u = \tilde{A} + \sqrt{\tilde{A}^2 - 1}.$$

This equation uniquely determines  $\tilde{A}$  as

$$\tilde{A} = \frac{e^{-\mu\Delta t} + e^{\mu\Delta t}}{2}.$$

Since  $\tilde{A} < A(\Delta t)$  for any  $\Delta t > 0$  it follows that the equation  $p(\Delta t) = 1$  has no solution for any  $\Delta t > 0$ . Hence  $p$  may be interpreted as a probability for all  $\Delta t$ .

With  $u$  and  $d$  determined we can now generate a binomial tree. Let  $0 = t_0 < t_1 < \dots < t_N = T$  with  $t_{n+1} - t_n = \Delta t$  be a partition of the interval  $[0, T]$ . Then starting with  $S_0^0$  we

can generate all the nodes  $S_k^n$ ,  $k = 0, \dots, n$  which can be reached by our discrete random walk by setting

$$S_{k+1}^{n+1} = uS_k^n, \quad S_{k+1}^{n+1} = dS_k^n.$$

The nodes at the time of expiry  $T$  are

$$S_k^N, \quad k = 0, \dots, N.$$

For example,  $S_0^N$  is reached if at all times the down branch of the tree is taken, so that the probability of reaching  $S_0^N$  is  $(1 - p)^N$ . The value of the option  $V_k^N$  at expiry is known at every node and provides the initial condition for recursively pricing the option at the nodes of the preceding time level. An arbitrage argument as in the Black Scholes model is used to find  $V_k^n$  from the option values at time  $t_{k+1}$ . Thus, suppose we have a portfolio at time  $t_n$  of the form

$$\pi = V - \Delta S$$

where  $\Delta$  is the number of shares of the underlying with value  $S$ . Let  $V^+$  and  $V^-$  be the option values at time  $t_{n+1}$  corresponding to  $uS$  and  $dS$ . Then the value of the portfolio will be the same at  $t_{n+1}$  if

$$V^+ - \Delta uS = V^- - \Delta dS$$

which implies

$$\Delta = \frac{V^+ - V^-}{(u - d)S}.$$

For a European option this value of the portfolio must be the same as the value of the riskless investment, i.e.

$$V^+ - \Delta uS = (V - \Delta S)e^{r\Delta t}.$$

A little algebra allows us to find  $V$  in terms of  $V^+$  and  $V^-$  as

$$V = e^{-r\Delta t}[\hat{p}V^+ + (1 - \hat{p})V^-]$$

where  $\hat{p} = \frac{e^{r\Delta t} - d}{u - d}$ . Note that  $\hat{p}$  depends on  $r$  and, through  $u$  and  $d$ , on the drift parameter  $\mu$ . If  $\hat{p} \in (0, 1)$  then  $\hat{p}$  may be thought of as a probability and the value of the option  $V_k^n$  at

$S_k^n$  is the discounted value of the expected value of the option at the nearest nodes at the next time level. The pricing formula is given by

$$V_k^n = e^{-r\Delta t} [\hat{p}V_{k+1}^{n+1} + (1 - \hat{p})V_k^{n+1}].$$

Hence we first simulate all possible binary random walks by building the binomial tree and then price the option at the nodes of the walk by moving backward through the tree from  $t = T$  to  $t = 0$ .

We note that the binomial tree reflects the stochastic movement of the underlying asset and is independent of the option. The pricing, however, depends on the option under consideration. The above formula holds for any European option. For an American option it must be augmented by the requirement that its values cannot fall below its intrinsic value. Hence the pricing formula is modified as in the explicit finite difference method by requiring that

$$P_k^n = \max \{K - S, e^{-r\Delta t} [\hat{p}P_{k+1}^{n+1} + (1 - \hat{p})P_k^{n+1}]\}$$

for a put, and

$$C_k^n = \max \{S - K, e^{-r\Delta t} [\hat{p}C_{k+1}^{n+1} + (1 - \hat{p})C_k^{n+1}]\}$$

for a call.

The pricing formula is a difference equation for the function  $V(S, t)$  at the three points  $(S, t - \Delta t)$ ,  $(uS, t)$  and  $(dS, t)$ . It is a common practice in numerical analysis to determine whether such a difference formula represents a consistent approximation to an underlying differential equation. The procedure for finding such differential equation is standard. Let  $\phi(S, t)$  be an arbitrary smooth function of  $S$  and  $t$ . In particular, we shall assume that

$$\max \left\{ \left| \frac{\partial^3 \phi}{\partial S^3} \right|, \left| \frac{\partial^2 \phi}{\partial t^2} \right| \right\} \leq K$$

for some constant  $K$  and all  $(S, t)$  in the domain where the difference formula is assumed

to hold. Then it follows from a Taylor expansion about the point  $(S, t)$  that

$$\begin{aligned}
& [\hat{p}\phi(uS, t) + (1 - \hat{p})\phi(dS, t)] - e^{r\Delta t}\phi(S, t - \Delta t) \\
&= \hat{p} \left[ \phi + \frac{\partial\phi}{\partial S}(uS - S) + \frac{1}{2} \frac{\partial^2\phi}{\partial S^2}(uS - S)^2 + \frac{1}{3!} \frac{\partial^3\phi(\tilde{S}, t)}{\partial S^3}(uS - S)^3 \right] \\
&\quad + (1 - \hat{p}) \left[ \phi + \frac{\partial\phi}{\partial S}(dS - S) + \frac{1}{2} \frac{\partial^2\phi}{\partial S^2}(dS - S)^2 + \frac{1}{3!} \frac{\partial^3\phi(\tilde{S}, t)}{\partial S^3}(dS - S)^3 \right] \\
&\quad - e^{r\Delta t} \left[ \phi - \frac{\partial\phi}{\partial t}\Delta t + \frac{1}{2} \frac{\partial^2\phi(S, \hat{t})}{\partial t^2}\Delta t^2 \right],
\end{aligned}$$

where  $\tilde{S}$ ,  $\tilde{S}$  and  $\hat{t}$  are some (unknown) intermediate points. Collecting coefficients we can rewrite this equation as

$$\begin{aligned}
& [1 - e^{r\Delta t}]\phi + [p(u - 1)S + (1 - p)(d - 1)S] \frac{\partial\phi}{\partial S} \\
&\quad + \frac{S^2}{2} [p(u - 1)^2 + (1 - p)(d - 1)^2] \frac{\partial^2\phi}{\partial S^2} \\
&\quad = O(\Delta t^2 + (u - 1)^3 + (d - 1)^3)
\end{aligned}$$

where the right hand side simply indicates that the remainder terms depend on the given rates. Now we observe that

$$e^{r\Delta t} = 1 + r\Delta t + O(\Delta t^2).$$

$$\hat{p}(u - 1) + (1 - \hat{p})(d - 1) = \hat{p}(u - d) + d - 1 = e^{r\Delta t} - 1 = r\Delta t + O(\Delta t^2)$$

and

$$\begin{aligned}
\hat{p}(u - 1)^2 + (1 - \hat{p})(d - 1)^2 &= \hat{p}(u - d)(u + d) - 2\hat{p}(u - d) + d^2 - 2d + 1 \\
&= (e^{r\Delta t} - d)(u + d - 2) + d^2 - 2d + 1 \\
&= e^{r\Delta t}(u + d - 2) - ud + 1.
\end{aligned}$$

At this stage we bring in the properties of  $u$  and  $d$  which up to now have not been used.

We see from a Taylor expansion that

$$(u + d - 2) = \left[ e^{-\mu\Delta t} + e^{(\mu + \sigma^2)\Delta t} - 2 \right] = \sigma^2\Delta t + O(\Delta t^2)$$

so that

$$e^{r\Delta t}(u + d - 2) = \sigma^2\Delta t + O(\Delta t^2).$$

Together with  $ud = 1$  it follows that

$$\hat{p}(u - 1)^2 + (1 - \hat{p})(d - 1)^2 = \sigma^2 \Delta t + O(\Delta t^2).$$

Dividing through by  $\Delta t$  we find that

$$\begin{aligned} & [\hat{p}\phi(uS, t) + (1 - \hat{p})\phi(dS, t)] - e^{r\Delta t}\phi(S, t - \Delta t) / \Delta t \\ & - \left[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \phi}{\partial S^2} + rS \frac{\partial \phi}{\partial S} - r\phi + \frac{\partial \phi}{\partial t} \right] = O(\Delta t + (u - 1)^3 / \Delta t + (d - 1)^3 / \Delta t). \end{aligned}$$

Finally, we observe that

$$u - 1 = \sqrt{A - 1} \left( \sqrt{A - 1} + \sqrt{A + 1} \right) = O\left(\sqrt{\Delta t}\right)$$

and

$$d - 1 = \sqrt{A - 1} \left( \sqrt{A - 1} - \sqrt{A + 1} \right) = O\left(\sqrt{\Delta t}\right)$$

so that the remainder term on the right goes to zero like  $\sqrt{\Delta t}$ . Hence the binomial formula is a consistent approximation to the Black-Scholes equation. We see that the drift parameter  $\mu$  does not appear in the Black-Scholes equation so that its solution only depends on the riskless interest rate  $r$ . Since in the limit as  $\Delta t \rightarrow 0$  the value of  $\mu$  drops out it is reasonable to generate the binomial tree with  $\mu = r$  which is an observable quantity. In this case  $\hat{p} = p \in (0, 1)$  is known as the risk-neutral probability.

Consistency of the approximation is no guarantee that the solution computed with the binomial method will converge to the solution of the Black-Scholes equation as  $\Delta t \rightarrow 0$ . However, convergence is simple to show for a European option by observing that for  $p \in (0, 1)$  the binomial algorithm is necessarily stable so that the Lax equivalence theorem applies. A direct argument can be made as follows.

Let  $V(S, t)$  be an analytic solution of the Black-Scholes equation with bounded higher derivatives, and let  $V_k^n$  be the value obtained from the binomial formula at the point  $S_k^n$  where  $S_k^n = dS_k^{n-1}$  and  $S_{k+1}^n = uS_k^{n-1}$ . Then the above analysis shows that the magnitude of the “error”

$$E_k^n = |V(S_k^n, t_n) - V_k^n|$$

satisfies

$$E_k^{n-1} \leq e^{-r\Delta t} [pE_{k+1}^n + (1 - p)E_k^n] + C\Delta t^{3/2}$$

where  $C$  depends on  $\frac{\partial^2 V}{\partial t^2}$  and  $\frac{\partial^3 V}{\partial S^3}$  but not on  $\Delta t$ . Since  $p \in (0, 1)$  we see that

$$\max_k E_k^{n-1} \leq \max_k E_k^n + C\Delta t^{3/2}.$$

From  $E_k^N = 0$  follows now

$$\max_k E_k^n \leq (N - n)C\Delta t^{3/2} < CT\Delta t^{1/2}.$$

Hence the error  $E_k^n$  goes to zero like  $\Delta t^{1/2}$ , i.e. the values obtained with the binomial method converge to the values of the Black-Scholes equation, albeit slowly like  $\Delta t^{1/2}$ .

The choice of  $u$  and  $d$  in the binomial method is not unique. For example

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = 1/u$$

is suggested in [Hull]. It is straightforward to verify that for this choice we again obtain a consistent approximation of the Black-Scholes equation and an unchanged rate of convergence provided that

$$\hat{p} = \frac{e^{r\Delta t} - d}{u - d} \in (0, 1).$$

It follows from

$$\hat{p} = \frac{e^{r\Delta t} - e^{\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}} + 1$$

and from

$$\lim_{\Delta t \rightarrow 0} \hat{p}(\Delta t) = \frac{1}{2}$$

that this can be the case only if

$$0 < \Delta t < \left(\frac{\sigma}{r}\right)^2$$

so that in this case the binomial method is subject to a time step constraint.

Another choice advocated in the literature is

$$u = e^{\mu\Delta t} \left(1 + \sqrt{e^{\sigma^2\Delta t} - 1}\right)$$

$$d = e^{\mu\Delta t} \left(1 - \sqrt{e^{\sigma^2\Delta t} - 1}\right)$$

Now  $ud \neq 1$ , but it is straightforward to verify that for this  $u$  and  $d$  the binomial method is again a consistent approximation of the Black-Scholes equation. With  $\mu = r$  we obtain convergence for all  $\Delta t$  since  $u$  and  $d$  were chosen such that

$$p = \hat{p} = \frac{1}{2}.$$

We conclude with the opinion that as a numerical method for the solution of the Black-Scholes equation the binomial method is not attractive since well chosen numerical methods have a higher rate of convergence. Its value must derive from the conviction that it models the actual market processes better than the time continuous Black-Scholes model.