MODULE 23

Topics: Cauchy's integral formula

Let Γ be a simple closed curve and suppose that f is analytic inside Γ and on Γ. Let us consider the function $g(z)$ defined by

$$g(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(s)ds}{(s-z)}$$

where z is an arbitrary but fixed point inside Γ. We can rewrite this function in the form

$$g(z) = \frac{1}{2\pi i} \left[\int_\Gamma \frac{f(z)}{s-z}\, ds + \int_\Gamma \frac{f(s) - f(z)}{s-z}\, ds \right].$$

Since

$$\int_\Gamma \frac{ds}{s-z}\, ds = 2\pi i$$

we see that

$$g(z) = f(z) + \frac{1}{2\pi i} \int_\Gamma \frac{f(s)-f(z)}{s-z}\, ds.$$

Cauchy's theorem allows us to shrink the contour Γ to a circle $|s-z| = \epsilon$ without changing the value of the integral. But as $\epsilon \to 0$

$$\frac{f(s)-f(z)}{s-z} \to f'(z)$$

so that

$$\left| \int_\Gamma \frac{f(s)-f(z)}{s-z}\, ds \right| \simeq |f'(z)| \left| \int_\Gamma ds \right| \leq |f'(z)| \int_{|z-s|=\epsilon} |ds| = |f'(z)| 2\pi\epsilon.$$

Since this has to hold for all ϵ no matter how small, it follows that

$$\int_\Gamma \frac{f(s)-f(z)}{s-z}\, ds = 0,$$

and we have

Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(s)}{s-z}\, ds.$$

where Γ is any simple closed curve around z and where f is analytic inside and on Γ. Formal differentiation under the integral shows that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int \frac{f(s)}{(s-z)^{n+1}}\, ds.$$

The argument can be made rigorous with an $\epsilon - \delta$ argument so that we can conclude: If f is analytic in a domain D then f is infinitely differentiable at any point of D. In particular, this implies that if f is analytic then u and v are harmonic without further hypotheses on the differentiability of the Cauchy-Riemann equations. A simple consequence of the formula for the derivatives is the following result. Suppose that on a circle of radius R around z the analytic function f is bounded, i.e.

$$|f(s)| \leq M \qquad \text{on } |s-z| = R$$

then

$$|f^{(n)}(z)| \leq \frac{n!M}{R^n}.$$

Suppose that f is analytic in the whole complex plane. Such a function is called an entire function. If, in addition, the function is also bounded by same constant M for all z then we may take an arbitrarily large circle in our estimate

$$|f'(z)| \leq \frac{M}{R}$$

and conclude that $f'(z) = 0$ for all z so that a bounded entire function must be a constant.

Fundamental Theorem of Algebra: *Every non-constant polynomial $P_N(z)$ has a root.*

Proof: We may write $P_N(z) = z^N(1 + a_{N-1}/z + \cdots + a_0/z^N)$ which shows that for $|z| > R_0$, where R_0 is sufficiently large, we can assert that $|P_N(z)| > |z|^N/2$. Let us suppose that $P_N(z) = 0$ does not have a solution inside the circle of radius R_0. Then the function

$$f(z) = \frac{1}{P_N(z)}$$

is differentiable and does not blow up inside the circle. Hence $|f(z)| < K$ for some constant K for all z in the circle. Since outside the circle

$$|f(z)| \leq \frac{2}{R_0^N}$$

it follows that $f(z)$ is entire and bounded in the whole complex plane. This would imply that f is constant, but this cannot be because by hypothesis, $P_N(z)$ is not a constant. It follows that $P_N(z)$ must have a root z_1 inside the circle. One can show by algebraic manipulation that for any integer k the expression $z^k - z_1^k$ can be factored into

$$z^k - z_1^k = (z - z_1^k)Q_{k-1}(z)$$

where $Q_{k-1}(z)$ is a polynomial of degree $k-1$. This allows us to "deflate" $P_N(z)$:

$$P_N(z) = P_N(z) - P_N(z_1) = (z - z_1)Q_{N-1}(z).$$

We then establish the existence of a root for Q_{N-1} and so on until we have N roots for P_N.

Cauchy's integral formula has many other important applications. For example, since

$$f(z) = u + iv = \frac{1}{2\pi i}\int_\Gamma \frac{f(s)}{s-z}\,ds,$$

it can be shown that the real part of the integral is a solution of

$$u_{xx} + u_{yy} = 0 \quad \text{inside } \Gamma$$

when u is a given function on Γ. Thus we have a formula for the solution of the potential problem in a simply connected domain D when the potential is given on the boundary of D. For a circle the famous Poisson formula results. Here we shall follow another tack and use Cauchy's integral formula to find a series representation for f.

Theorem: *Let f be analytic in the annulus $D = \{z : r < |z - z_0| < R\}$ then we can expand f in terms of a so-called Laurent series*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \qquad \text{for all } z \in D.$$

Proof: Let z be an arbitrary point in D. Let Γ_1 and Γ_2 be two circles around z_0 inside the annulus with z in between them. If we integrate around z along the path indicated in class then it follows from Cauchy's integral formula that

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(s)}{s-z}\,ds - \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(s)}{s-z}\,ds.$$

We can formally rewrite this identity as

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(s)}{(s-z_0)\left(1-\dfrac{z-z_0}{s-z_0}\right)}\,ds - \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(s)}{(z-z_0)\left(\dfrac{s-z_0}{z-z_0}-1\right)}\,ds.$$

On Γ_2: $|(z-z_0)/(s-z_0)| < 1$ and on Γ_1 we have $|(s-z_0)/(z-z_0)| < 1$ so that on Γ_2:

$$\frac{1}{1-\left(\dfrac{z-z_0}{s-z_0}\right)} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{s-z_0}\right)^n$$

and on Γ_1:

$$\frac{1}{1-\left(\dfrac{s-z_0}{z-z_0}\right)} = \sum_{n=0}^{\infty}\left(\frac{s-z_0}{z-z_0}\right)^n.$$

Assuming that integration and summation can be interchanged we see that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

where

$$a_k = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(s)}{(s-z_0)^{k-1}}\,ds \qquad \text{for } k = 0, 1, 2, \dots$$

and

$$a_{-k} = \frac{1}{2\pi i}\int_{\Gamma_1} f(s)(s-z_0)^k\,ds \qquad \text{for } k = 1, 2, 3, \dots$$

The infinite series is known as a Laurent series for the function f. In practice, the series is rarely found by actually integrating around the indicated circles. Rather, the existence of such a series is important and the basis for actual calculations. However, we do observe that if $f(z)$ is analytic everywhere inside and on the larger contour Γ_2 then

$$a_{-k} = 0 \quad \text{for } k = 1, 2, \dots$$

and, by the Cauchy integral formula,

$$a_k = \frac{f^{(k)}(z_0)}{k!} \qquad \text{for } k = 0, 1, 2,$$

so that the Laurent series becomes the Taylor series for f. In particular, the result implies that the Taylor series for f around z_0 has a radius of convergence equal to the distance from z_0 to the nearest singularity of f.