On the Derivation and Numerical Solution of the Black Scholes Barenblatt Equation for Jump Diffusion

Gunter H. Meyer
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA
meyer@math.gatech.edu

Abstract

The classical maximum principle argument is used to derive the Black Scholes Barenblatt Partial Integro-Differential Equation which yields bounds for option prices of a jump diffusion process with uncertain volatility. The nonlinear partial differential equation is solved as a sequence of time discrete ordinary integro-differential equations by applying a simple fixed point iteration at each time level. The algorithm requires the repeated solution of a Black Scholes equation with a source term depending on the jump integral and the convexity of the solution. Its solution is obtained from the Riccati transformation for which the jump integral is approximated by Gaussian quadrature.

1. Introduction

When the value S of a financial asset is modeled with the stochastic differential equation for geometric Brownian motion

$$dS = \mu S\, d\tau + \sigma(S,\tau) S\, dW \tag{1.1}$$

then the price V of an option written on the asset and expiring after T years is governed by the so-called Black Scholes equation

$$\mathcal{L}(\sigma)u \equiv \frac{1}{2}\,\sigma^2(x,K,t)x^2 u_{xx} + rxu_x - ru - u_t = 0. \tag{1.2}$$

Here $x = S/K$ and $u = V/K$ for some reference price K which is chosen to be the strike price of the option. μ is the drift rate of the asset and r is the prevailing risk-free interest rate. Both rates may depend on x and t. $t = T - \tau$ for calendar time τ is the time to expiration which is chosen as independent variable to obtain the usual forward diffusion equation treated in textbooks on partial differential equations and their numerical solution.

The coefficient σ in (1.1) and (1.2) is the volatility of $S(t)$ which is of central concern in this investigation. For notational convenience and emphasis we denote the Black Scholes differential operator in (1.2) by $\mathcal{L}(\sigma)$. Equation (1.2) is augmented with initial and boundary conditions which are specific to the option that is to be priced.

The Black Scholes equation (1.2) implies that the local volatility surface $\sigma(x, K, t)$, henceforth written as $\sigma(x, t)$, is a deterministic function of the scaled price x and t. Whether the volatility is indeed a deterministic function, and if so, what its form should be, constitute two dominant questions for the application of (1.2). In practice the volatility is not known with certainty, but one often can impose reasonable bounds $\sigma_0(x,t), \sigma_1(x,t)$ such that

$$\sigma_0(x,t) \leq \sigma(x,t) \leq \sigma_1(x,t). \tag{1.3}$$

The Black Scholes Barenblatt equation (BSB equation) can then be used to obtain attainable lower and upper bounds on the solution of the Black Scholes equation (1.2) for any admissible volatility satisfying (1.3).

One version of the Black Scholes Barenblatt equations associated with (1.2), (1.3) is the fully nonlinear generalization of (1.2)

$$\mathcal{L}(\overline{\sigma})u = \pm\frac{1}{2}\,\underline{\sigma}^2\,x^2|u_{xx}| \equiv \pm F(x,t,|u_{xx}|) \tag{1.4}$$

where $\overline{\sigma}^2 = \frac{\sigma_1^2+\sigma_0^2}{2}$ and $\underline{\sigma}^2 = \frac{\sigma_1^2-\sigma_0^2}{2}$. The solutions $u_0(x,t)$ of (1.4) with source term $+F$ and $u_1(x,t)$ with source term $-F$ subject to the boundary and initial conditions of (1.2) yield bounds on the solution of (1.2) such that

$$u_0(x,t) \leq u(x,t) \leq u_1(x,t)$$

for all admissible volatility functions $\sigma(x,t)$ satifying (1.3), with equality holding for some particular choice of $\sigma(x,t)$ in this range. The class of admissible volatilities will be specified below when the derivation of the BSB equation is discussed.

To illustrate the kind of results obtainable with the Black Scholes Barenblatt equation suppose that the equation (1.1) describes a CEV process

$$dx = \mu x \, d\sigma + \widehat{\sigma} \left(\frac{x}{\widehat{x}}\right)^{\gamma} x \, dW$$

where $\widehat{\sigma}$ is a given (implied) volatility at the reference point $\widehat{x}$. γ is the elasticity of the process and is in general not known with certainty. We wish to examine the influence of the elasticity γ on the value of an option when we assume that

$$\underline{\gamma} \le \gamma \le \overline{\gamma} \tag{1.5}$$

for given limiting values $\underline{\gamma}$ and $\overline{\gamma}$. We write

$$\sigma_0(x) = \widehat{\sigma} \min \left\{ \left(\frac{x}{\widehat{x}}\right)^{\underline{\gamma}}, \left(\frac{x}{\widehat{x}}\right)^{\overline{\gamma}} \right\}$$
$$\sigma_1(x) = \widehat{\sigma} \max \left\{ \left(\frac{x}{\widehat{x}}\right)^{\underline{\gamma}}, \left(\frac{x}{\widehat{x}}\right)^{\overline{\gamma}} \right\}$$

so that

$$\sigma_0(x) \le \sigma(x) \le \sigma_1(x)$$

for any $\gamma \in [\underline{\gamma}, \overline{\gamma}]$.

Figs. 1.1a,b show computed bounds u_0 and u_1 as well as two CEV option prices at two different times for a European straddle with a down and out barrier at $X_0 = .8$ and an up and out barrier at $X_1 = 1.2$. Since K is the strike price of the option the equations (1.2) and (1.4) are subject to the initial and boundary data

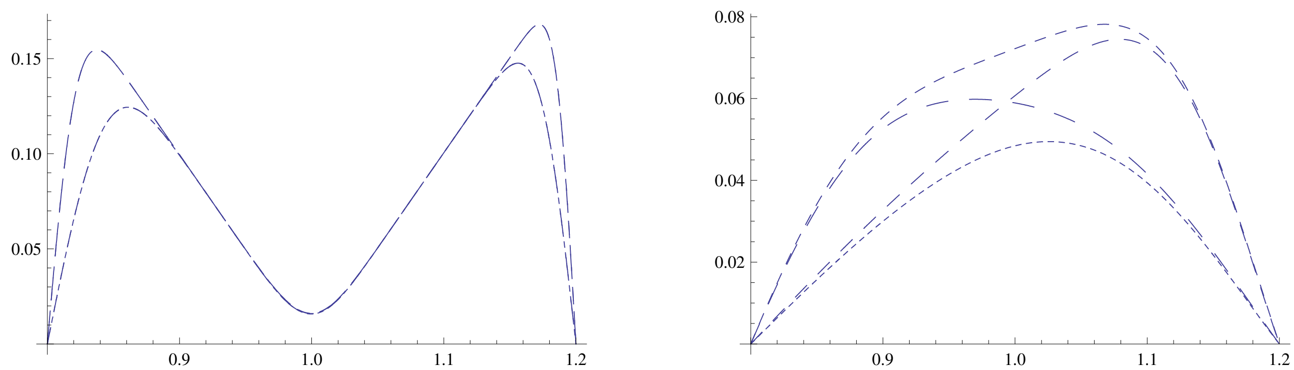
$$u(x, 0) = \max\{1 - x, 0\} + \max\{0, x - 1\}$$

$$u(.8, t) = u(1.2, t) = 0.$$

The displayed bounds correspond to $K = \widehat{x} = 1$ and

$$\underline{\gamma} = -4, \qquad \overline{\gamma} = -.5.$$

The two Black Scholes option prices hold for the CEV straddle with elasticity $\gamma = -4$ and $\gamma = -.5$.



a) at $t = .01$, b) at $t = .3$.

Fig. 1.1 Upper and lower bounds on attainable option prices for a European double barrier staddle for a CEV process with uncertain elasticity:

Some numerical values for the plotted solutions are

$$u_0(.84, .01) = .10999, \qquad u_1(.84, .01) = .15427$$

$$U_0(.84, .01) = .10999, \qquad U_1(.84, .01) = .15427$$

$$u_0(.84, .3) = .01264, \qquad u_1(.84, .3) = .02826$$

$$U_0(.84, .3) = .01353, \qquad U_1(.84, .3) = .02742$$

where

$u_0 =$ BSB lower bound

$u_1 =$ BSB upper bound

$U_0 =$ Black Scholes solution for elasticity $\gamma = -4$

$U_1 =$ Black Scholes solution for elasticity $\gamma = -.5$.

As we shall see below, the upper and lower bounds correspond to a Black Scholes solution for volatility functions which are discontinuous at points where the Black Scholes Barenblatt solution changes convexity. Hence the bounds are not assumed for a CEV process which necessarily has a smooth volatility. Nonetheless, Fig. 1.1a shows that the Black Scholes prices locally can almost coincide with the BSB bounds.

It is the aim of this exposition to show that the BSB formalism of [9] remains unchanged when the equation (1.1) is extended to a jump diffusion process

$$dx = \mu x \, d\tau + \sigma(x, \tau) x \, dW + (J - 1) x \, dq \tag{1.6}$$

where random jumps from x to Jx are modeled with a Poisson process in time. We shall also demonstrate that the same simple iterative numerical algorithm used in [8] for pricing options on assets with jumps, and in [9] for finding the BSB bounds on assets following (1.1), remains effective for computing BSB bounds when the asset price follows the jump diffusion model (1.6) with an uncertain but bounded volatility.

2. Derivation of the Black Scholes Barenblatt equation for jump diffusion

We assume that the value S of an asset follows the jump diffusion model

$$dS = (\mu - \rho)S\,d\tau + \sigma(S,\tau)S\,dW + (J-1)S\,dq \tag{2.1}$$

where μ is the drift rate of the asset and $\rho \geq 0$ is the annualized rate at which the asset pays a dividend. As before $\sigma(S,\tau)$ is an uncertain but bounded volatility function and $W(\tau)$ denotes a Wiener process. $(J-1)S\,dq$ stands for random jumps from S to JS in the value of the asset distributed according to a probability density function $p(J)$. The jumps constitute a Poisson process in time with intensity λ. As before we shall write our equations in terms of $t = T - \tau$, $x = S/K$ and $u(x,t) = V(Kx,t)/K$.

When only diffusion is hedged then the pricing equation for an option written on this asset is the following generalization of the Black Scholes equation [12]

$$\begin{aligned} \mathcal{L}(\sigma)u \equiv \frac{1}{2}\sigma^2(x,t)x^2u_{xx} + (r-\rho-\lambda k)xu_x - ru \\ + \lambda\int_0^\infty (u(Jx,t) - u(x,t))p(J)dJ - u_t = 0 \end{aligned} \tag{2.2}$$

where

$$k = \int_0^\infty (J-1)p(J)dJ$$

and $r > 0$ is the risk-free interest rate. Henceforth $\mathcal{L}(\sigma)$ denotes the Black Scholes jump diffusion operator defined by equation (2.2).

Equation (2.2) is called a partial-integro-differential equation (PIDE). It is augmented with an initial condition

$$u(x,0) = u_0(x) \tag{2.3a}$$

and boundary conditions which are specific to the option to be priced. We shall write them for European options in generic form as barrier conditions

$$\begin{aligned} u(X_0, t) &= f_0(t) \\ u(X_1, t) &= f_1(t) \end{aligned} \tag{2.3b}$$

and set

$$\begin{aligned} u(x, t) &= f_0(t), \qquad x \in [0, X_0] \\ u(x, t) &= f_1(t), \qquad x \geq X_1 \end{aligned}$$

so that the integrals in (2.2) are defined. We assume that the initial/boundary data are continuous, and as before, that

$$0 < \sigma_0(x, t) \leq \sigma(x, t) \leq \sigma_1(x, t)$$

where σ_0 and σ_1 are given lower and upper bounds on the uncertain volatility σ. Admissible volatilities, dividend rates, interest rates and jump distributions are those functions for which the problem (2.2), (2.3a,b) has a classical solution on $[X_0, X_1] \times [0, T]$.

Suppose now that u is twice continuously differentiable with respect to x and once differentiable with respect to t, and that it satisfies the inequality

$$\mathcal{L}(\sigma)u \geq 0 \qquad \text{for } x \in (X_0, X_1), \quad t \in (0, T] \tag{2.4}$$

and the homogeneous initial and boundary conditions

$$u_0 = f_0 = f_1 \equiv 0.$$

Then standard maximum principle arguments rule out that u attains a positive global maximum [2]. Indeed, if

$$u(x^*, t^*) > 0 \quad \text{for some } (x^*, t^*) \in (X_0, X_1) \times (0, T]$$

and

$$u(x^*, t^*) \geq u(x, t) \quad \text{for all } (x, t)$$

then

$$u_{xx}(x^*, t^*) \leq 0, \quad u_x(x^*, t^*) = 0, \quad u_t(x^*, t^*) \geq 0$$

and

$$u(Jx^*, t^*) - u(x^*, t^*) \leq 0 \quad \text{for all } J \in (0, \infty)$$

because u is set to zero outside $[X_0, X_1]$. It follows that

$$\mathcal{L}(\sigma)u(x^*, t^*) \leq -ru(x^*, t^*) < 0$$

contrary to assumption (2.4). Similarly, if

$$\mathcal{L}(\sigma)u \leq 0$$

then there cannot be a negative minimum.

The Black Scholes Barenblatt equation for jump diffusion can now be derived with the arguments employed in [9].

Let u be a classical solution of (2.2), (2.3). Let u_0 be a classical solution of the nonlinear PIDE

$$\mathcal{L}(\sigma)u_0 = \frac{1}{2}(\sigma^2(x,t) - \sigma_0^2(x,t))x^2 u_{0xx}^+ + \frac{1}{2}(\sigma^2(x,t) - \sigma_1^2(x,t))x^2 u_{0xx}^- \tag{2.5}$$

subject to the initial and boundary conditions imposed on u, where $\sigma(x,t)$ is any admissible volatility in the prescribed band, and where $a^+ = \max\{a, 0), a^- = \min\{a, 0\}$. We assume that such a classical solution exists. Then by inspection

$$\mathcal{L}(\sigma)(u - u_0) = -\mathcal{L}(\sigma)u_0 \leq 0.$$

It follows that $u - u_0$ has no negative minimum and hence that

$$u_0 \leq u.$$

Similarly, if u_1 is the solution of

$$\mathcal{L}(\sigma)u_1 = \frac{1}{2}(\sigma^2(x,t) - \sigma_0^2(x,t))x^2 u_{1xx}^- + \frac{1}{2}(\sigma^2(x,t) - \sigma_1^2(x,t))x^2 u_{1xx}^+ \tag{2.6}$$

subject to the initial and boundary data of u, then

$$\mathcal{L}(\sigma)(u - u_1) = -\mathcal{L}(\sigma)u_1 \geq 0.$$

The maximum principle then assures that $u - u_1$ has no positive global maximum so that

$$u - u_1 \leq 0 \quad \text{everywhere.}$$

Hence for arbitrary $\sigma(x,t)$ in the admissible band we have upper and lower bounds

$$u_0(x,t) \leq u(x,t) \leq u_1(x,t).$$

At first glance the equations (2.5), (2.6) seem unsolvable since the unknown volatility $\sigma(x,t)$ appears in them. But in fact, σ will cancel out which leads to equivalent equations depending on $\sigma_0(x,t)$ and $\sigma_1(x,t)$ as shown below. Equations (2.5), (2.6) are useful because they show that u_0 and u_1 are bounds. In addition (2.6) implies that an option written on x can be superreplicated. To see this consider a self-financing portfolio π in real time τ

$$\pi(x(\tau),\tau) = u_1(x,\tau) - \Delta x(\tau) - B(\tau), \quad \Delta = u_{1x}(x(\tau),\tau) \tag{2.7}$$

such that $\pi(x(0),0) = 0$. We know from [5, p. 39] that

$$d\pi = du_1 - \Delta\, dx - rB(\tau)d\tau$$

so that from [12, p. 328] with delta hedging

$$d\pi = \left[\left(u_{1\tau} + \frac{1}{2}\sigma^2 x^2 u_{1xx}\right) + \lambda(u_1(Jx,\tau) - u_1(x,\tau) - \Delta(J-1)x)\right] d\tau - rB(\tau)d\tau.$$

Taking expectations over all jumps we obtain

$$d\pi = \left[\left(u_{1\tau} + \frac{1}{2}\sigma^2 x^2 u_{1xx}\right) - \lambda k x u_{1x} + \lambda \int_0^\infty (u_1(Jx,\tau) - u_1(x,\tau))p(J)dJ\right] d\tau - rB(\tau)d\tau.$$

Using (2.7) to eliminate $B(\tau)$ and comparing with the BSB equation (2.6) we find that

$$d\pi(x(\tau),\tau) = r\pi\, d\tau + [h(\tau)]d\tau, \quad \pi(x(0),0) = 0 \tag{2.8}$$

where $h(\tau)$ stands for the source term of equation (2.6). Since $h(\tau) \leq 0$ it follows that the solution of (2.8) satisfies $\pi(x(\tau),\tau) \leq 0$, and in particular that at expiration $\tau = T$

$$u_0(x) = u_1(x(T),T) \leq u_{1x}(x(T),T)x(T) + B(T)$$

for any payoff $u_0(x)$. Thus the option payoff u_0 is superreplicated at $\tau = T$.

We now observe that in equation (2.5) the terms involving the unknown volatility cancel so that (2.5) can be rewritten as the nonlinear PIDE

$$\mathcal{L}(f_0(u_{0xx}))u_0 = 0 \tag{2.9}$$

where

$$f_0(u_{xx}) = \begin{cases} \sigma_0(x,t) & \text{if } u_{xx} \geq 0 \\ \sigma_1(x,t) & \text{if } u_{xx} < 0. \end{cases}$$

In other words, (2.9) depends only on the data of the problem. Similarly, equation (2.6) can be written in the form

$$\mathcal{L}(f_1(u_{1xx}))u_1 = 0 \tag{2.10}$$

where

$$f_1(u_{xx}) = \begin{cases} \sigma_0(x,t) & \text{if } u_{xx} \leq 0 \\ \sigma_1(x,t) & \text{if } u_{xx} > 0. \end{cases}$$

The PIDEs (2.9), (2.10) are the generalizations of the well known Black Scholes Barenblatt equations associated with the Black Scholes equation where $\lambda = 0$.

Equations (2.9), (2.10) show that the bounds u_0 and u_1 are sharp (i.e., attainable). Indeed, if we can solve the nonlinear problem (2.5) or (2.9) for u_0 then the corresponding volatility

$$\sigma(x,t) = f_0(u_{0xx})$$

in the jump diffusion model (2.2) yields a price u_0 which coincides with the lower bound on all prices for volatilities in the prescribed band. Similarly, if we can find u_1 then for the volatility

$$\sigma(x,t) = f_1(u_{1xx})$$

the jump diffusion price u_1 is a sharp upper bound on all prices with volatilities in the prescribed band.

When the solutions of (2.9) and (2.10) do not change convexity then the equations are no longer nonlinear but reduce to the standard linear pricing equation for jump diffusion. However, when the solution of the PIDE changes convexity then the volatilities f_0 and

f_1 are discontinuous and it is not obvious that the corresponding solutions u_0 and u_1 are smooth. We shall comment on this point again below.

For analysis and numerical work it is convenient to rewrite equations (2.5), (2.6) as a standard jump diffusion pricing equation with a nonlinear source term. With the substitution

$$u_{xx}^{+} = \frac{u_{xx} + |u_{xx}|}{2}, \quad u_{xx}^{-} = \frac{u_{xx} - |u_{xx}|}{2}$$

equation (2.5) becomes

$$\mathcal{L}(\overline{\sigma}(x,t))u_0 = F(x,t,u_{0xx}) \tag{2.11}$$

while (2.6) becomes

$$\mathcal{L}(\overline{\sigma}(x,t))u_1 = -F(x,t,u_{1xx}) \tag{2.12}$$

where

$$\overline{\sigma}^2(x,t) = \frac{\sigma_0^2(x,t) + \sigma_1^2(x,t)}{2}$$

and

$$F(x,t,u_{xx}) = \frac{1}{2}\,\frac{\sigma_1^2(x,t) - \sigma_0^2(x,t)}{2}\,x^2|u_{xx}| \tag{2.13}$$

We note that if w is a solution of (2.11) subject to the negative of the initial and boundary conditions given for u then

$$u_1 = -w.$$

We also observe that the source term (2.13) has the same structure as the source term for the Black Scholes equation which accounts for proportional transaction costs [9].

In summary, the PIDE (2.5), (2.9), (2.11) and (2.6), (2.10), (2.12) are algebraically equivalent and will all be referred to as BSB equations for a jump diffusion process.

3. The numerical solution of the Black Scholes Barenblatt PIDE

The time discrete method of lines algorithm used in [9] for the solution of the Black Scholes Barenblatt equation without jumps and in [8] for pricing options with the Black Scholes equation for jump diffusion will be applied here to solve the Black Scholes Barenblatt PIDE (2.11). Conceptually the method consists of an approximation of the PIDE by an ordinary integro-differential equation (OIDE) at a discrete time level t_n and a solution method for the OIDE in the underlying variable x.

Experience with the method of lines approach to option pricing strongly suggests the need for second order accuracy in approximating the time derivative u_t. On the basis of simplicity and performance we favor a fully implicit backward Euler scheme for the first three time levels and a change-over to a simple three level scheme for subsequent time levels. For ease of exposition we choose a constant time step

$$\Delta t = \frac{T}{N}, \qquad N \geq 1$$

and set

$$t_n = n\Delta t, \quad n = 0, 1, \ldots, N.$$

Then we write the approximation

$$u_t(x, t_n) \cong D_n u = \begin{cases} \frac{u(x) - u_{n-1}(x)}{\Delta t} & n = 1, 2, 3 \\ \frac{3}{2}\frac{u(x) - u_{n-1}(x)}{\Delta t} - \frac{1}{2}\frac{u_{n-1}(x) - u_{n-2}(x)}{\Delta t} & n \geq 4. \end{cases}$$

where $u(x)(\equiv u_n(x))$ is the solution of the OIDE

$$\frac{1}{2}\overline{\sigma}^2 x^2 u'' + (r - \rho - \lambda k)xu' - (r + \lambda)u - D_n u = -\lambda \int_0^\infty u(Jx)p(J)dJ + \frac{1}{2}\underline{\sigma}^2 x^2 |u''|$$

subject to the boundary conditions imposed on u at time t_n. We shall rewrite this equation in the form

$$L_n u \equiv \frac{1}{2}\overline{\sigma}^2 x^2 u'' + (r - \rho - \lambda k)xu' - (r + \lambda + c_n)u = H(x, u) \tag{3.1}$$

where

$$c_n = \begin{cases} \frac{1}{\Delta t} & n = 1, 2, 3 \\ \frac{3}{2\Delta t} & n \geq 4. \end{cases}$$

$$H_n(x, u) = -\lambda \int_0^\infty u(Jx)p(J)dJ + \frac{1}{2}\underline{\sigma}^2 x^2 |u''| + h_n(x),$$

and where

$$h_n(x) = D_n u - c_n u$$

provides the link to the preceding time levels.

The nonlinear equation (3.1) is fully time implicit in u and will be solved with the simple substitution iteration

$$L_n u^k = H_n(x, u^{k-1}), \qquad k = 1, 2, \ldots \tag{3.2}$$

$$u^k(X_0) = f_0(t_n), \quad u^k(X_1) = f_1(t_n)$$

where u^0 is an initial guess. We usually set $u^0(x) = u_{n-1}(x)$.

The iteration (3.2) requires the repeated solution of a linear two point boundary value problem. It is conceptually simple and, as shown in [8], advocated even for the linear jump diffusion problem which results when $\underline{\sigma} = 0$. However, any numerical solution of (3.2) will have to contend with all the difficulties brought into the formulation by the nonlocal nature of the source H_n.

We remark that in the case of pure jump diffusion, i.e., $\underline{\sigma} = 0$, the iteration (3.2) is identical to the iteration used in [3] and [1] where the Crank-Nicolson method is used for the space and time discretization. In [3] the integral is transformed into a correlation integral and approximated with FFT methods. An alternate method for evaluating the integral is suggested in [1] where for commonly used exponentially decaying probability density functions the integral is recognized to be a representation formula for the solution of a Cauchy problem for the heat equation. The values of the integral can then be found from a numerical solution of the Cauchy problem. We refer to [1] for a discussion of the relative merits of these two approaches.

Our numerical solution of (3.2) will be based on a standard quadrature of the integral terms and the Riccati method for linear two point boundary value problems. The advantage of this approach lies in its simplicity and generality. Whether quadrature as implemented below can compete with the treatment of the integral in [3] and [1] is not clear, but the sample calculations below give an indication that this approach is feasible.

Details of the method of lines discretization coupled with a Riccati sweep method have been given repeatedly for European and American options. (see, for example, [7] and [8]). In the Riccati method only initial value problems for ordinary differential equations are integrated with the trapezoidal rule. Initial value methods typically are insensitive to the placing of grid points which can be exploited by calculating the integral in the source term $\{H_n(x, u^{k-1}(x))\}$ without interpolation by using u^{k-1} only at the grid points $\{x_j\}$ chosen for the trapezoidal rule. We shall proceed as follows.

The integral is written in the form

$$\int_0^\infty u(Jx)p(J)dJ = \frac{1}{x}\int_0^\infty u(y)g(y/x)dy = A_{(0,X_0)}(x) + A_{(X_0,X_1)}(x) + A_{(X_1,X_\infty)}(x)$$

where

$$A_{(a,b)}(x) = \frac{1}{x}\int_a^b u(y)p(y/x).$$

$A_{(0,X_0)}(x)$ and $A_{(X_1,X_\infty)}(x)$ are readily found because u is given analytically outside the interval $[X_0, X_1]$, or $u(y)p(y/x)/x$ is essentially zero there. $A_{(X_0,X_1)}(x)$ requires numerical quadrature for each grid point x_j. A straightforward and reasonably accurate numerical method results when we combine the integration of the differential equations with a composite two-point Gaussian quadrature for each $A_{(X_0,X_1)}(x_j)$. To this end we define first auxiliary grid points $\{z_i\}$ with $X_0 = z_0 < z_1 < \cdots < z_I = X_1$ on $[X_0, X_1]$. Their selection is guided by the usual considerations which apply to a non-adaptive gridding of a computational domain for an ODE solver. For example, $\{z_i\}$ will bunch up near barriers and near discontinuities of the source term of the equation where the solution is expected to have large gradients. The grid points $\{x_j\}$ for the actual calculation consist of the end-points X_0 and X_1 and, for each interval $[z_{i-1}, z_i]$, of the two Gaussian quadrature points

$$\frac{(z_i + z_{i-1} \pm 1/\sqrt{3}(z_i - z_{i-1})]}{2}.$$

Specifically, we shall find u at the grid points $\{x_j\}$, $j = 0, \ldots, \tilde{J}$ where

$$x_0 = X_0,$$

where for $i = 1, \ldots, I$

$$j = 2i$$

$$x_{j-1} = \frac{(z_i + z_{i-1} - 1/\sqrt{3}(z_i - z_{i-1})]}{2}$$

$$x_j = \frac{(z_i + z_{i-1} + 1/\sqrt{3}(z_i - z_{i-1})]}{2}$$

and where

$$x_{\tilde{J}} = X_1 \quad \text{for } \tilde{J} = 2I + 1.$$

The integral $A_{(X_0,X_1)}(x_j)$ is then approximated by the composite formula

$$A_{(X_0,X_1)}(x_j) = \sum_{i=1}^{I} A_{(z_{i-1},z_i)}(x_j) = \sum_{i=1}^{I} \frac{(z_i - z_{i-1})}{2x_j} \left[u(x_{2i-1}) p\left(\frac{x_{2i-1}}{x_j}\right) + u(x_{2i}) p\left(\frac{x_{2i}}{x_j}\right) \right].$$

Once $\{A_{(X_0,X_1)}(x_j)\}$ is found for all j the source term $H(x_j, u^{k-1}(x_j))$ is available. It is now straighforward to integrate the sweep equations arising from the Riccati method for the boundary value problem (3.2) with the trapezoidal rule as described in detail in [7]. The iteration is terminated when

$$\max_j |u^k(x_j) - u^{k-1}(x_j)| \leq \epsilon$$

for a specified tolerance ϵ.

We know that for a continuous source term the numerical solution of the two point boundary value problem (3.2) obtained with the Riccati transformation converges to the analytic solution of (3.2) because the resulting initial value problems can be stably integrated. The method remains useful for the Black Scholes Barenblatt PIDE because after a few time steps the simple substitution iteration (3.2) converges sufficiently fast as $k \to \infty$ to accept the computational cost associated with the numerical quadrature required for each iterate u^k at each mesh point x_j.

To illustrate the feasibility of this approach let us apply the above algorithm to a combination of an American put and a European call with an up and out barrier at $X_1 = 2$. This "strangle-like" instrument is not claimed to be significant financially. It is chosen solely to introduce the numerical complications of loss of convexity and a free boundary into the problem.

The following boundary and initial conditions describe this option:

$$\begin{aligned} u(s(t), t) &= 1 - s(t) \\ u_x(s(t), t) &= -1 \\ u(2, t) &= 0 \\ u(x, 0) &= \max\{0, 1 - x\} + \max\{0, x - 1\}. \end{aligned} \tag{3.3}$$

The initial and boundary conditions are discontinuous at $(x,t) = (2,0)$ and need to be smoothed in order to obtain a classical solution. For the time discrete approximation we shall assume that the smoothing occurred over time $t < \Delta t$ so that it does not enter into the numerical calculation.

In order to have a well defined jump integral we shall set

$$u(x,t) = 1 - x \qquad \text{for } x \in (0, s(t))$$

$$u(x,t) = 0 \qquad \text{for } x > 2.$$

For simplicity we shall assume that the jumps are uniformly distributed according to

$$p(J) = \begin{cases} 1 & .8 < J < 1.8 \\ 0 & \text{otherwise} \end{cases}$$

so that

$$k = E[J-1] = .3.$$

Again, we do not claim that this density is meaningful in finance although a uniform density has been used before to fit market data [4]. Here it is chosen to introduce uncertainty into how the free boundary will behave near expiration.

We note that a uniform jump density has only marginal influence on the efficiency of the calculation compared to a more common exponentially decaying probability density function because the density $p(x_i/x_j)$ is computed at the mesh points only once and then stored for subsequent use in each iteration and at every new time step. For the remaining parameters we choose

$$\lambda = .1, \quad r = .05, \quad \rho = 0, \quad T = .1 \quad \text{and} \quad \Delta t = .1/100.$$

The upper and lower bounds to be found with the BSB PIDE correspond to the volatility range

$$\sigma_0 = .1 \le \sigma \le \sigma_1 = .5.$$

Fig. 3.1 shows the upper bound u_1 and the lower bound u_0 as well as the actual option price for a fixed volatility of

$$\sigma = .3.$$

Fig. 3.2 shows the free boundaries $s_1(t)$ and $s_0(t)$ associated with u_1 and u_0 as well as the early exercise boundary $s(t)$ for $\sigma = .3$. It is known [9] that in the absence of jumps, i.e. $\lambda = 0$,

$$s_1(t) \leq s(t) \leq s_0(t).$$

We have no proof of these inequalities for jump diffusion but expect them, and observe them to hold, in the present case as well.

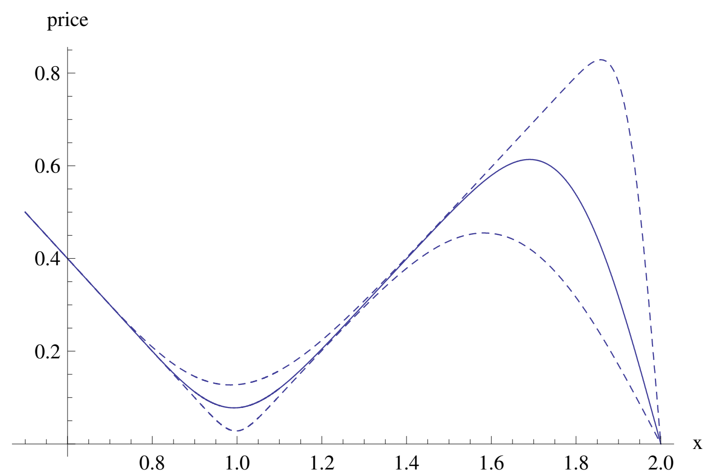


Fig. 3.1 Black Scholes Barenblatt solutions u_0 and u_1 for $\sigma_0 = .1$ and $\sigma_1 = .5$, and intermediate Black Scholes jump diffusion solution u for $\sigma = .3$ at $t = .1$. $\Delta t = .1/100$, 3200 spatial integration points $\{x_j\}$ on $[.5, 2]$; convergence tolerance $\epsilon = 10^{-6}$.

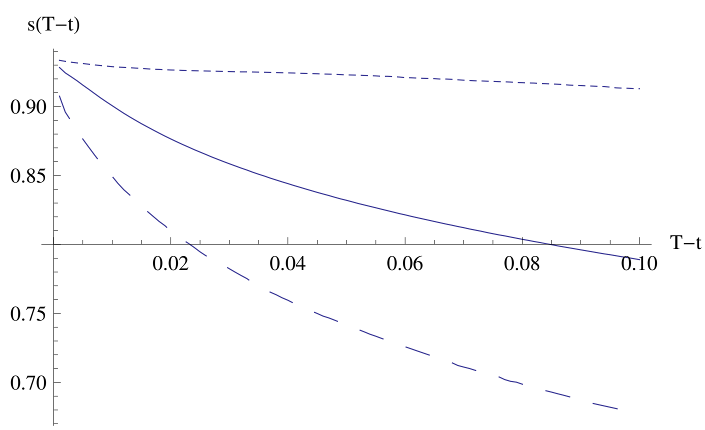


Fig. 3.2 Free boundaries $s_0(t)$ and $s_1(t)$ corresponding to $u_0(x, t)$ and $u_1(x, t)$ and early exercise boundary $s(t)$ for $u(x, t)$ at $t = .1$.

We would like to point out that for the option and jump distribution considered here we find from the computed data that $\lim_{t\to 0} s(t) \neq 1$. This is apparent from Table 3.1 which lists the position of the free boundary $s(\Delta t)$ one time step before expiration for a sequence of decreasing Δt.

Table 3.1 Convergence of $s(\Delta t)$ as $\Delta t \to 0$ for the jump diffusion problem (2.2), (3.3)

Δt	$s(\Delta t)$
.01	.8762
.001	.9282
.0001	.9355
.00001	.9349

$\lambda = .1, \quad r = .05, \quad \rho = 0, \quad \sigma_0 = \sigma = \sigma_1 = .3$

The resolution in x is sufficiently fine so that these computed values essentially reflect the error due to time differencing only.

A "back of the envelope calculation" gives some insight into this behavior of $s(t)$ as $t \to 0$. Assuming a smooth solution of the jump diffusion equation we know from

$$u(s(t), t) = 1 - s(t)$$
$$u_x(s(t), t) = -1$$

that $u_t(s(t), t) \equiv 0$ for $t > 0$. It follows from the PIDE that at $t = \Delta t$

$$\frac{1}{2}\sigma^2 s^2 u_{xx}(s+, t) = (\lambda k + \rho - r)s(-1) + (r + \lambda)(1 - s) - \lambda \int_0^\infty u(J(1-s), t)p(J)dJ \quad (3.4)$$

In order for $u(x, t)$ not to fall below the intrinsic value $\max\{1 - x, 0\}$ (i.e. to remain on or above the obstacle) we need $u_{xx}(s+) \geq 0$. We do not know $u(x, t)$ but expect that

$$\lim_{t\to 0} \int_0^\infty u(Jx, t)p(J)dJ = \int_0^\infty u_0(Jx)p(J)dJ, \quad x \in [0, 2].$$

For the pay-off $u_0(x) = \max\{1 - x, 0\} + \max\{0, x - 1\}$ for $x \in [0, 2]$ and the uniform jump distribution chosen above the latter integral is easy to evaluate. We find from

$$\int_0^\infty u_0(Js)p(J)dJ = \int_{.8}^{1/s} [1 - Js]dJ + \int_{1/s}^{1.8} [Js - 1]dJ = \frac{1}{s} + 1.94s - 2.6.$$

It follows that

$$\frac{1}{2}\sigma^2 s^2 u_{xx}(s+,t) = -(\rho + \lambda k + 2.94\lambda)s - \frac{\lambda}{s} + (r + 3.6\lambda).$$

For $\rho = 0$, $k = .3$ and $\lambda = .1$ we obtain $u_{xx}(s+,t) \geq 0$ provided that $s \in [.3299, .9355]$.

While this calculation does not assert that $s(t) \to .9355$ it does show that our numerical free boundary is consistent with the analytic (approximate) bound. (For a rigorous analysis of the behavior of the free boundary near expiry for an American put with jump diffusion we refer to [10].)

It is generally observed that unless the bounds on σ are very tight the spread on the BSB prices $u_1 - u_0$ is too large to be of practical value. The spread can sometimes be reduced through a technique called static hedging (for a recent discussion see, e.g. [9]). Static hedging applies equally in the context of jump diffusion. In particular, put-call parity for European calls and puts remains in effect. The portfolio

$$\pi(x,t) = p(x,t) - c(x,t)$$

satisfies the linear jump diffusion equation (2.2) and the initial condition

$$\pi(x,0) = 1 - x.$$

The corresponding BSB PIDEs for $\pi_0(x,t)$ and $\pi_1(x,t)$ are assumed to have unique solutions, and it is straightforward to verify that

$$\pi_0(x,t) = \pi_1(x,t) = e^{-rt} - x$$

satisfies (2.11) and (2.13). Hence if the call is traded today then todays price of the put is uniquely determined regardless of the value of σ. On the other hand, if the traded instruments have different expiration, strike prices and are subject to different barriers then static hedging in the presence of jump diffusion becomes a challenge for numerical methods.

4. Convergence

Throughout the preceding sections we have assumed that the BSB PIDE for the double barrier strangle has a unique classical solution. Published results for the BSB equation with Cauchy data [11] and for linear integro-differential equations [2] give confidence that our problem does indeed have a classical solution so that the BSB PIDE is meaningful. Here we shall be concerned only with the convergence of the iterative solution of the time discrete problem. For ease of notation we shall henceforth set $\rho = 0$.

Let $\mathcal{M}$ denote the set of twice continuously differentiable functions of the Hilbert space $W^{2,2}[X_0, X_1]$ of functions with square integrable second derivatives which take on the constant values $f_0(t_n)$ on $[0, X_0)$ and $f_1(t_n)$ on (X_1, ∞).

For given f and $g \in \mathcal{M}$ let $u = Tf$ and $v = Tg$ denote the solutions of

$$L_n u = H_n(x, f)$$

and

$$L_n v = H_n(x, g)$$

subject to the boundary data of (3.2). We may assume that $h_n(x)$ is continuous so that $H_n(x, f)$ and $H_n(x, g)$ are continuous source terms for the two point boundary value problem (3.2). It follows that u and v are classical solutions so that T maps $\mathcal{M}$ into $\mathcal{M}$. Our goal is to establish first that T is contractive on $W^{2,2}[X_0, X_1]$. With the notation $w = u - v$, $\langle f, g \rangle = \int_{X_0}^{X_1} f(x)g(x)dx$ and $\|f\| = \sqrt{\langle f, f \rangle}$ we have

$$\langle L_n w, w'' \rangle = \langle H_n(x, f) - H_n(x, g), w'' \rangle \tag{4.1}$$

For any $\epsilon > 0$ the algebraic geometric mean inequality yields

$$|(r - \lambda k)\langle xw', w'' \rangle \leq |r - \lambda k| \left[\frac{\langle w', w' \rangle}{2\epsilon} + \epsilon \frac{\langle xw'', xw'' \rangle}{2} \right].$$

Similarly, since $f - g = 0$ outside the interval $[X_0, X_1]$ we obtain

$$\begin{aligned} | \left\langle \int_0^\infty [f(Jx) - g(Jx)]p(J)dJ, w'' \right\rangle | &= | \int_0^\infty \left\langle \frac{f(Jx) - g(Jx)}{x}, xw'' \right\rangle p(J)dJ| \\ &\leq \frac{k+1}{2\epsilon X_0^2} \langle f - g, f - g \rangle + \frac{\epsilon}{2} \langle xw'', xw'' \rangle. \end{aligned}$$

It now follows from (4.1) that

$$\begin{aligned}\left[\frac{1}{2}\overline{\sigma}^2 - \frac{1}{4}\underline{\sigma}^2 - \epsilon\,\frac{|r-\lambda k|+1}{2}\right]\langle xw'', xw''\rangle + \left[r+\lambda+c_n - \frac{|r-\lambda k}{2\epsilon}\right]\langle w', w'\rangle \\ \leq \frac{k+1}{2\epsilon x_0^2}\,\langle f-g, f-g\rangle + \frac{1}{4}\underline{\sigma}^2\langle x(f''-g''), x(f''-g'')\rangle.\end{aligned} \tag{4.2}$$

Since also

$$\langle w, w\rangle \leq \left(\frac{X_1 - X_0}{\pi}\right)^2 \langle w', w'\rangle$$

we can obtain from (4.2) the inequality

$$A_1\langle xw'', xw''\rangle + A_2\langle w, w\rangle \leq A_3\langle x(f''-g''), x(f''-g'')\rangle + A_4\langle f-g, f-g\rangle \tag{4.3}$$

where

$$\begin{aligned} A_1 &= \left[\frac{1}{2}\,\overline{\sigma}^2 - \frac{1}{4}\,\underline{\sigma}^2 - \epsilon\frac{|r-\lambda k|+1}{2}\right], \\ A_2 &= \left[r+\lambda+c_n - \frac{|r-\lambda k|}{2\epsilon}\right]\left(\frac{\pi}{X_1-X_0}\right)^2, \\ A_3 &= \frac{1}{4}\,\underline{\sigma}^2 \quad \text{and} \quad A_4 = \frac{k+1}{2\epsilon X_0^2}\,. \end{aligned}$$

For $\sigma_0 < \sigma_1$ we can pick ϵ so small that $A_3 < A_1$. We can now choose Δt so small (i.e. c_n so large) that $A_4 < A_2$.

Then with the notation

$$\langle\langle u, v\rangle\rangle = A_1\langle xu'', xv''\rangle + A_2\langle u, v\rangle \quad \text{and} \quad |||u|||^2 = \langle\langle u, u\rangle\rangle$$

we see from (4.3) that

$$|||u-v|||^2 = |||Tf - Tg|||^2 \leq \gamma^2 |||f-g|||^2$$

where

$$\gamma^2 = \max\left[\frac{A_3}{A_1}, \frac{A_4}{A_2}\right] < 1. \tag{4.4}$$

It follows that T is a contraction on $\mathcal{M}$ and that the fixed point iteration $u^k = Tu^{k-1}$ converges in $W^{2,2}[X_0, X_1]$ to a fixed point u^* of T from any $u^0 \in \mathcal{M}$.

For the remainder of this section we shall assume that u^0 is chosen such that

$$|u''^0(x) - u''^0(y)| < K\sqrt{|x-y|}, \quad \text{and that} \quad |h_n(x) - h_n(y)| < K\sqrt{|x-y|}$$

for some constant K. We shall prove that u''^* is Holder continuous on $[X_0, X_1]$.

It follows from $|||u^k||| \le |||u^0||| + \sum_{j=1}^k |||u^j - u^{j-1}|||$ that

$$|||u^k||| \le |||u^0||| + \frac{1}{1-\gamma}\,|||u^1 - u^0|||$$

so that

$$\langle u^k, u^k\rangle, \quad \langle xu''^k, xu''^k\rangle \quad \text{and hence} \quad \langle u'^k, u'^k\rangle$$

are uniformly bounded. These integral bounds imply pointwise bounds. Let M denote a generic constant. Then

$$|u^k(x)| \le |f(t_n)| + C\|u'^k\| \le M \quad \text{for some } M \text{ and all } x \in [X_0, X_1].$$

Next we observe that

$$xu'(x) = (xu(x))' - u(x) = (xu(x))'_{x=\alpha} + \int_\alpha^x (su(s))''ds - u(x)$$

where for given ξ, η such that $|\eta - \xi| \ge (X_1 - X_0)/3$ the point α is such that

$$\frac{\eta u(\eta) - \xi u(\xi)}{\eta - \xi} = (xu(x))'_{x=\alpha}.$$

Thus for any x

$$xu'(x) = \frac{\eta u(\eta) - \xi u(\xi)}{\eta - \xi} + \int_\alpha^x su''(s)ds - 2u(\alpha)$$

which implies that there are constants K and M such that for all k

$$|xu'^k(x)| \le K(\max|u^k(x)| + \|xu''^k\|) \le M.$$

The OIDE

$$\frac{1}{2}\,\overline{\sigma}^2 x^2 u''^k(x) = \frac{1}{2}\underline{\sigma}^2 x^2 |u''^{k-1}(x)| - (r - \lambda k)xu'^k(x) + (r + \lambda + c_n)u^k(x) \\ - \lambda \int_0^\infty u^{k-1}(Jx)p(J)dJ + h_n(x),$$

and the pointwise bounds on u^k and u'^k allow the conclusion that

$$\frac{1}{2}\overline{\sigma}^2|x^2u''^k(x)| \leq \frac{1}{2}\,\underline{\sigma}^2|x^2u''^{k-1}(x)| + M$$

for some constant M which is independent of the iteration number k. Since $\underline{\sigma} < \overline{\sigma}$ it follows that $|x^2u''^k(x)|$ also is uniformly bounded for all x and k. Next we observe that the uniform boundedness of $\|u'^k\|$ and $\|xu''^k\|$ imply that

$$|u^k(x) - u^k(y)| = |\int_y^x u'^k(s)ds| \leq M\sqrt{|x-y|}$$

$$|xu'^k(x) - yu'^k(y)| = |\int_y^x (su'^k(s))'ds| \leq M\sqrt{|x-y|}$$

and

$$|\int_0^\infty [u^k(Jx) - u^k(Jy)]p(J)dJ| \leq M\sqrt{|x-y|}\int_0^\infty Jp(J)dJ \leq M\sqrt{|x-y|}.$$

It now follows from the OIDE that

$$\frac{1}{2}\overline{\sigma}^2|x^2u''^k(x) - y^2u''^k(y)| \leq \frac{1}{2}\,\underline{\sigma}^2|x^2u''^{k-1}(x) - y^2u''^{k-1}(y)| + M\sqrt{|x-y|}$$

so that there exists a constant M such that

$$|x^2u''^k(x) - y^2u''^k(y)| \leq M\sqrt{|x-y|}$$

for all k. Thus $\{x^2u''^k(x)\}$ is a bounded equicontinuous family of functions on $[X_0, X_1]$ and every convergent subsequence of $\{u^k\}$ converges to an element $u^* \in \mathcal{M}$. Since T is contractive on $\mathcal{M}$ there cannot be distinct fixed points in $\mathcal{M}$. Hence for every smooth initial guess u^0 the fixed point iteration will converge to a unique solution $u^* \in \mathcal{M}$ of the OIDE (3.1).

We do not assert that these results hold uniformly at all time levels, and we do not know whether the technique associated with the name of "Rothe's method" (see, e.g. [6, p. 241]) can be used to prove convergence of the time discrete solutions to a solution of the time dependent PIDE.

We shall conclude with some comments on the observed and theoretical rate of convergence for the double barrier strangle analyzed above. For the numerical simulation we shall assume the initial and boundary data

$$u(0,t) = \max\{1 - x, 0\} + \max\{0, x - 1\}$$
$$u(.5,t) = u(2,t) = 0$$

and the financial and computational parameters of problem (2.2), (3.3) i.e.

$$\lambda = .1, \quad r = .05, \quad \rho = 0, \quad \sigma_0 = .1, \quad \sigma_1 = .5, \quad T = .1 \quad \text{and} \quad \Delta t = .1/100.$$

Again, the spatial resolution is sufficiently fine that the numerical solution of the time discrete problem (3.2) is thought to be very accurate.

Fig. 4.1 shows a plot of the number of iterations necessary at each time step so that u_0 and u_1 satisfy

$$\max_j |u^k(x_j) - u^{k-1}(x_j)| \leq 10^{-6}.$$

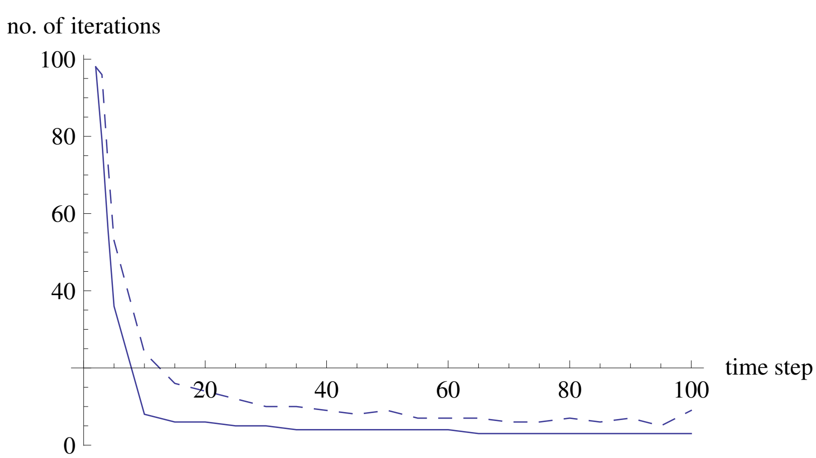


Fig. 4.1 Observed rate of convergence for the iteration (3.2)

—— number of iterations for convergence to u_0

- - - number of iterations for convergence to u_1.

It is apparent that the observed rate of convergence initially is much the same for u_0 and u_1 but that in general u_1 is harder to compute than u_0. Reducing Δt has no noticeable effect on the required number of iterations.

The a priori estimate (4.4), which is only sufficient for convergence, is consistent with the observed convergence behavior. For the parameters of the calculation we find that

$$\frac{A_3}{A_1} = \frac{.12}{.14 - 2.04\epsilon}\,.$$

Hence for the admissible $\epsilon \in (0, .02/2.04)$ we obtain a contraction constant $\gamma \geq \sqrt{6/7}$ regardless of the choice of Δt. In view of this large contraction constant the high number of iterations observed for the first few time steps and its independence of Δt are not surprising.

In contrast, for $\sigma_0 = \sigma = \sigma_1 = .3$ we obtain the following iteration count at the first time level

Table 4.1 Iteration count at time $t = \Delta t$ for the jump diffusion PIDE (2.2)

t	no. of iterations
1.0	6
.5	5
.1	4
$\leq .01$	3

Thus the slow initial convergence is due entirely to the nonlinear term $|u_{xx}|$. Fortunately, once the solution settles down only few iterations per time step are required which make the fixed point iteration (3.2) a feasible choice for the solution of the BSB PIDE (2.11), (2.12). Whether an algorithm based on the BSB PIDE (2.9), (2.10) is more efficient remains to be examined.

References

1. P. Carr and A. Mayo, On the numerical evaluation of option prices in jump diffusion processes, *The European J. Finance* **13** (2007), 353–372.
2. M. G. Garroni and J. L. Menaldi, *Second Order Elliptic Integro-Differential Problems*, Res. Notes Math. 430, Chapman & Hall,/CRC, 2002.
3. Y. d'Halluin, P. A. Forsyth and K. R. Vetzal, Numerical methods for contingent claims under jump diffusion processes, *IMA Journal Num. Anal.* **25** (2005), 87–112.

4. F. B. Hanson and J. J. Westman, Jump-diffusion stock return models in finance: Stochastic process density with uniform jump amplitude, www.math.uic.edu/∼hanson/pub/MTNS2002/mtns02fmt.pdf
5. Y. K. Kwok, *Mathematical Models of Financial Derivatives*, Springer Verlag, 1998, ISBN 9813083255.
6. O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Uralceva, Linear and Quasilinear Equations of Parabolic Type, *Transl. Math. Monographs*, Vol. 23, AMS, 1968.
7. G. H. Meyer and J. van der Hoek, The valuation of American options with the method of lines, *Advances in Futures and Options Research* **9** (1997), 265–285.
8. G. H. Meyer, The numerical valuation of options with underlying jumps, *Acta Math. Univ. Comenianae* **67** (1998), 69–82.
9. G. H. Meyer, The Black Scholes Barenblatt equation for options with uncertain volatility and its application to static hedging, *Int. J. Theoretical Applied Finance* **9** (2006), 673–703.
10. Huyen Pham, Optimal stopping, free boundary, and American option in a jump diffusion model, *Appl. Mathematics Optimization* **35** (1997), 145–164.
11. T. Vargiolu, Existence, uniqueness and smoothness for the Black Scholes Barenblatt equation, www.math.unipd.it/∼vargiolu/BSB.ps
12. P. Wilmott, *Derivatives*, Wiley and Sons, 1998/1999, ISBN 0-471-98389-6.