The Black Scholes Barenblatt Equation for Options with Uncertain Volatility and its Application to Static Hedging

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Working paper, October 2004

Abstract

The Black Scholes Barenblatt (BSB) equation for the envelope of option prices with uncertain volatility and interest rate is derived from the Black Scholes equation with the maximum principle for diffusion equations and shown to be equivalent to a readily solvable standard Black Scholes equation with a nonlinear source term. Analogous arguments yield the envelope for the delta of option prices. We then interpret the concept of static hedging for narrowing the envelope in terms of partial differential equations and define the optimal static hedge and computable approximations to it. We apply the BSB equation to find numerically some optimally hedged portfolios of representative European and American options.

1. Introduction

The price $V(S,t)$ of a financial option for buying or selling an asset of value S is generally found from the Black Scholes equation

$$\mathcal{L}V \equiv \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV - V_t = 0$$

where t denotes the time to expiry of the option, i.e., $t = T - \tau$ where T is the time of expiry and τ is calendar time. This equation is solved subject to initial data at $t = 0$ and boundary data which are specific to the option to be considered.

In the Black Scholes equation occur two parameters (or functions), the volatility σ of the evolution of the price S of the underlying asset with time, and the risk-free interest rate r. Both quantities influence the price $V(S,t)$ but they are not known with certainty. In particular, the volatility is a troublesome input. Whether a single numerical value, a deterministic function implied by market data or a stochastic probability (with an attendant change in the above Black Scholes equation) should be chosen remains an open question and subject of intense debate and research (see, e.g., [11], [5], [7]). The risk-free interest rate is somewhat easier to pin down since it is closely linked to the spot rate, but for long-term options a term structure for r is appropriate which also is not known with certainty.

While the choice for σ and r may be controversial, upper and lower bounds on the volatility and interest during the life of the option can often be imposed with reasonable certainty. Such bounds, which may depend on S and t as well, allow us to compute the envelopes* within which the option price has to fall. These envelopes give a quantitative measure of the exposure of the option price to changes in the volatility and the interest rate. This information is not obtainable from the usual Greeks

$$\frac{\partial V}{\partial \sigma}, \frac{\partial V}{\partial r}$$

because the envelopes will be shown to be Black Scholes solutions for discontinuous volatility and interest rate functions for which these Greeks are not defined.

For plain vanilla puts and calls the option prices are known to be monotone and convex. In this case the envelopes are also described by a linear Black Scholes equation. However, for options with non-monotone and non-convex value at any time the envelopes must be found from a fully nonlinear equation of the form

$$\mathcal{L}V = F(S, t, V, V_S, |V_{SS}|).$$

In the context of finance this equation is known as the Black Scholes Barenblatt (BSB) equation. As a rule it can be solved only numerically.

In the literature the BSB equation is derived with hedging and arbitrage arguments parallel to those leading to the Black Scholes equation. It is the purpose of this paper to show that the BSB equation follows from the Black Scholes equation. Our arguments are based on an application of the maximum principle for the diffusion equation and apply in principle to all diffusion equations with uncertain but bounded coefficients. As we shall see, there are three equivalent forms of the BSB equation, each with their own advantages for

* The term "envelopes" is used here to denote achievable least upper and greatest lower bounds on all admissble option prices.

analysis and computation. One of these forms is identical in structure to the Black Scholes type equation which incorporates transaction costs. This equation is straightforward to solve numerically.

Upper and lower bounds on option prices are useful to insure that options sold are not underpriced and options bought are not overpriced for any volatility and interest rate function within the stated range. When these bounds are too conservative they often can be improved through static hedging and applying the BSB theory to a suitable portfolio of options. In the second part of the paper we define and analyze static hedging in a PDE context and show how the bounds can be tightened through optimal static hedging or an approximation to it.

2. Derivation of the envelope equations for the option price

To be specific we shall consider first a simple up and out European call $V(S,t)$ with expiration T, strike price K and an up and out barrier at $S = X > K$. As is well known, the price of this call is non-convex which introduces genuine nonlinearities into the problem.

We shall assume that for $S \in (0, X)$ and $t \in (0, T]$ the Black Scholes equation

$$\mathcal{L}(\sigma, r)V \equiv \frac{1}{2}\sigma^2(S,t)S^2V_{SS} + r(S,t)SV_S - r(S,t)V - V_t = 0 \tag{2.1}$$

applies, where σ and r are deterministic but uncertain volatility and interest rate functions. Boundary and initial conditions for the up and out call are

$$V(0,t) = V(X,t) = 0$$

$$V(S,0) = (S-K)^+$$

where a^+ and a^- stand for $\max\{a, 0\}$ and $\min\{a, 0\}$, respectively.

Since

$$\lim_{S \to X} V(S,0) \neq \lim_{t \to 0} V(X,t)$$

the call will have a discontinuity at $(X, 0)$. We shall avoid this complication by replacing the initial condition with the piecewise linear approximation

$$V(S,0) = \begin{cases} 0, & S < K \\ S - K, & K \leq S \leq (X - \epsilon) \\ (X - \epsilon - K)\,\frac{X-S}{\epsilon}\,, & X - \epsilon < S < X \end{cases}$$

where ϵ is small. We may assume that this problem has a unique classical solution $V(S,t)$. Since all our results will be independent of ϵ they will apply to the solution of the original problem which is also assumed to exist and to be smooth for $t > 0$.

While σ and r are not known with certainty we shall suppose that we have upper and lower bounds

$$\sigma_0(S,t) \leq \sigma(S,t)) \leq \sigma_1(S,t)$$

with $\sigma_0(S,t) \geq c > 0$, and

$$r_0(S,t) \leq r(S,t) \leq r_1(S,t)$$

with $r_0(S,t) \geq 0$. Our goal is to find functions $V_0(S,t)$ and $V_1(S,t)$ such that

$$V_0(S,t) \leq V(S,t) \leq V_1(S,t)$$

for all σ and r falling within the above bounds, with equality holding for a specific choice of σ and r. We assume throughout that (2.1) has a solution for all such σ and r.

The governing equations for V_0 and V_1 are derived in the literature by considering a best and worst case scenario for the value of a portfolio as the volatility and interest rates are allowed to vary freely within their assigned ranges [3], [4], [12]. However, they are already implied by the Black Scholes equation (2.1) and the standard maximum principle for parabolic equations. We recall that if a function $V(S,t)$ satisfies

$$\mathcal{L}(\sigma,r)V \leq 0 \quad \text{on } D = (0,X) \times (0,T],$$

where $\mathcal{L}(\sigma,r)$ is the operator defined by (2.1), and is continuous on $D = [0,X] \times [0,T]$, then V cannot have a negative minimum in D. If V has a negative minimum at all, then it must occur on the boundary of D, i.e., at $S = 0$, $S = X$ or at $t = 0$. Similarly, $\mathcal{L}(\sigma,r)V \geq 0$ in D rules out a positive maximum in D.

Let us consider now the equation

$$\begin{aligned} \mathcal{L}(\sigma,r)V_0 = \frac{1}{2}\, S^2[(\sigma^2 - \sigma_0^2)V_{0SS}^+ + (\sigma^2 - \sigma_1^2)V_{0SS}^- \\ + [(r - r_0)(SV_{0S} - V_0)^+ + (r - r_1)(SV_0 - V_0)^-] \end{aligned} \tag{2.2}$$

subject to the initial and boundary conditions imposed on the above call. $\mathcal{L}(\sigma, r)$ is the Black Scholes operator defined by equation (2.1).

We see that equation (2.2) is of the form

$$\mathcal{L}V_0 = F_0(S, V_0, V_{0S}, V_{0SS}) \tag{2.3}$$

where F_0 stands for the right hand side of (2.2). By inspection

$$F_0(S, V, V_S, V_{SS}) \geq 0.$$

Assume that V_0 is a solution of (2.2) on $(0, T] \times (0, X)$ which is continuous on $[0, T] \times [0, X]$ and takes on the initial and boundary conditions of the up and out call. If we set

$$e(S, t) = V_0(S, t) - V(S, t)$$

then

$$\mathcal{L}(\sigma, r)e \geq 0 \quad \text{ on } D$$

and, by construction,

$$e = 0$$

on $S = 0$, $S = X$ and on $t = 0$. By the maximum principle the function e cannot have a positive maximum in D. Non-positive boundary data then assure that

$$e(S, t) \leq 0 \quad \text{ in } D$$

which implies that

$$V_0(S, t) \leq V(S, t)$$

so that V_0 is a lower bound on the option price for any functions σ and r between the imposed limits.

If in equation (2.2) we interchange the maximum and minimum functions occurring in F (i.e., $V_{SS}^{+} \to V_{SS}^{-}$ etc.) and label the new equation

$$\mathcal{L}V_1 = F_1(S, V_1, V_{1S}, V_{1SS}) \tag{2.4}$$

then $F_1(S,V,V_S,V_{SS}) \le 0$. A solution of (2.4) subject to the boundary and initial conditions for the up and out call satisfies

$$\mathcal{L}(\sigma,r)(V_1 - V) \le 0$$

and by the maximum principle

$$V(S,t) \le V_1(S,t).$$

We observe that equation (2.2) can be rewritten in the form given in [3]. The terms involving σ and r always cancel out and we are left with

$$\mathcal{L}_0^{\mathrm{BSB}} V_0 \equiv \frac{1}{2} f_0(\sigma)^2 S^2 V_{0SS} + g_0(r)(SV_{0S} - V_0) - V_{0t} = 0 \tag{2.5}$$

where

$$f_0(\sigma) = \begin{cases} \sigma_0(S,t) & \text{if } V_{0SS} \ge 0 \\ \sigma_1(S,t) & \text{if } V_{0SS} < 0 \end{cases}$$

and

$$g_0(r) = \begin{cases} r_0(S,t) & \text{if } (SV_{0S} - V_0) \ge 0 \\ r_1(S,t) & \text{if } (SV_{0S} - V_0) < 0 \end{cases}.$$

We note that equation (2.5) implies that $V_0(S,t) \ge 0$ because if $V_0(S,t)$ has a negative minimum at some $(S^*,t^*) \in (0,X) \times (0,T]$ then

$$V_{0SS}(S^*,t^*) \ge 0, \quad V_{0S}(S^*,t^*) = 0, \quad V_{0t}(S^*,t^*) \le 0.$$

But these inequalities together with $V_0(S^*,t^*) < 0$ are inconsistent with equation (2.5).

Similarly, we obtain for the upper bound the equation

$$\mathcal{L}_1^{\mathrm{BSB}} V_1 \equiv \frac{1}{2} f_1(\sigma)^2 S^2 V_{1SS} + g_1(r)(SV_{1S} - V_1) - V_{1t} = 0 \tag{2.6}$$

where

$$f_1(\sigma) = \begin{cases} \sigma_0(S,t) & \text{if } V_{1SS} \le 0 \\ \sigma_1(S,t) & \text{if } V_{1SS} > 0 \end{cases}$$

and

$$g_1(r) = \begin{cases} r_0(S,t) & \text{if } (SV_{1S} - V_1) \le 0 \\ r_1(S,t) & \text{if } (SV_{1S} - V_1) > 0. \end{cases}$$

In the literature the equations (2.5) and (2.6) are called Black Scholes Barenblatt equations (BSB) (see, e.g., [3]).

A more symmetric form of these equations is obtained if we use the identities

$$a^+ = (a + |a|)/2, \qquad a^- = (a - |a|)/2$$

in (2.2) and (2.4).

Let us define the averages

$$\bar{\sigma}^2 = \frac{\sigma_0^2 + \sigma_1^2}{2}, \qquad \bar{r} = \frac{r_0 + r_1}{2},$$

and the function

$$\bar{F}(S, u, u_S, |u_{SS}|) = \frac{1}{2}\frac{\sigma_1^2 - \sigma_0^2}{2} S^2 |u_{SS}| + \frac{r_1 - r_0}{2} |S u_S - u|$$

then equations (2.2) and (2.4) can be rewritten as

$$\mathcal{L}(\bar{\sigma}, \bar{r}) V_0 = \bar{F}(S, V_0, V_{0S}, |V_{0SS}|) \tag{2.7}$$

$$\mathcal{L}(\bar{\sigma}, \bar{r}) V_1 = -\bar{F}(S, V_1, V_{1S}, |V_{1SS}|). \tag{2.8}$$

They must be solved subject to the initial and boundary condition

$$V_0(S, 0) = V_1(S, 0) = V(S, 0)$$

$$\begin{aligned} V_0(0, t) = V_1(0, t) = V(0, t) = 0 \\ V_0(X, t) = V_1(X, t) = V(0, t) = 0. \end{aligned} \tag{2.9}$$

Note that if we set

$$w(S, t) = -V_1(S, t) \tag{2.10}$$

then w is a solution of (2.7) subject to the initial condition

$$w(S, 0) = -V(S, 0).$$

Hence to find V_0 and V_1 we have the option of solving (2.7) and (2.8) subject to the same initial condition or (2.7) twice subject to initial conditions $V(S,0)$ and $-V(S,0)$. This observation extends to non-zero boundary data as long as V_0 and V_1 assume the same values on the boundary. The existence and uniqueness of a classical solution of (2.7) subject to the smoothed initial and boundary conditions is a question of mathematical analysis and will not be pursued here. We shall simply assume that V_0 and V_1 exist and allow an application of the maximum principle. The extensive numerical experiments outlined below support this assumption.

The equations {(2.3), (2.5), (2.7)} and {(2.4), (2.6), (2.8)} are all equivalent. They will all be called Black Scholes Barenblatt equations in this paper. They may be invoked to characterize various properties of their solutions. For example, the equations (2.2), (2.4) very simply yield that their solutions provide bounds on $V(S,t)$. The equations (2.5), (2.6) allow us to argue that we can find a volatility and interest rate function for which the Black Scholes solution coincides with the envelope, although in general these coefficients will be discontinuous at points where V changes convexity. Finally, the equations (2.7), (2.8) show that their nonlinear source terms are continuous as long as second derivatives are continuous which should allow an application of the linear existence and uniqueness theory for parabolic equations to an iterative solution of (2.7). But perhaps more importantly, these equations have the same structure as the non-linear Black Scholes equation for option trading with transaction cost due to Leland. We already know that this equation can be solved numerically by a convergent simple iteration [8] (see Appendix).

As an illustration we show in Fig. 2.1 the envelopes for a double barrier European straddle $V(S,t)$ for $t = T = 0.1$ described by

$$\mathcal{L}(\sigma, r)V(S,t) = 0$$

$$V(S,0) = (100 - S)^+ + (100 - S)^+ \tag{2.11}$$

$$V(80,t) = V(120,t) = 0$$

when

$$.1 < \sigma < .2$$

$$.05 < r < .06.$$

The curve between the envelopes is the solution of the straddle at T for the fixed choice $\sigma = .1$ and $r = .055$.

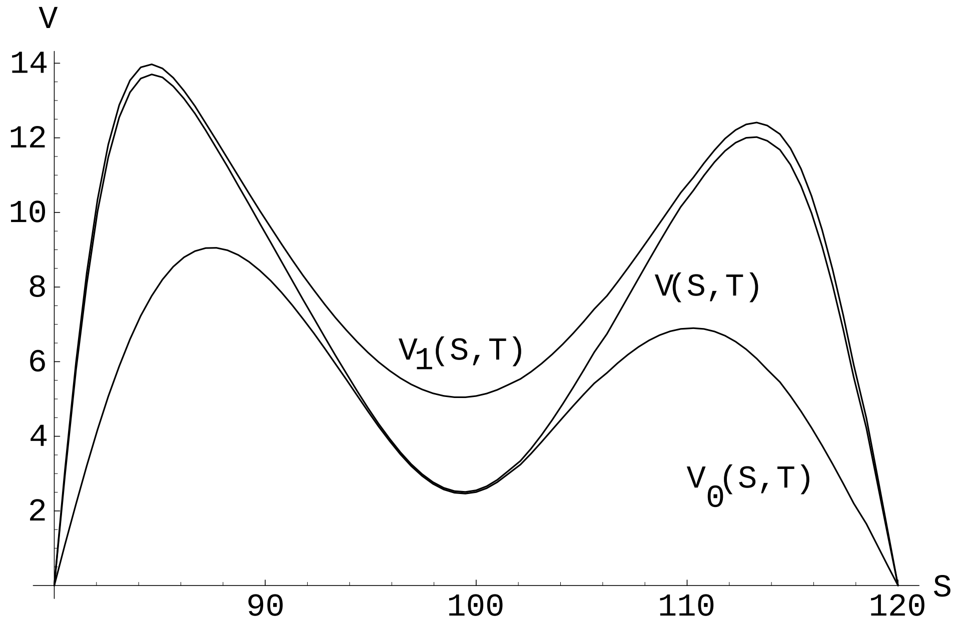

Fig. 2.1. Envelopes V_0, V_1 and Black Scholes Solution V for the European Double Barrier Straddle (2.11).

We observe from Fig. 2.1 that the volatility functions determined by V_0 and V_1 have jumps where the convexity changes, while the graph of V shows that even a constant volatility solution of the Black Scholes equation can locally approach both envelopes.

We note that the above derivations remain unchanged if the option is subject to boundary data like

$$V(B_\ell(t), t) = h_\ell(\sigma, r, t), \quad V(B_u(t), t) = h_u(\sigma, r, t)$$

for time dependent lower and upper barriers $B_1(t)$ and $B_u(t)$. In this case V_0 is subject to the greatest lower bound on the possible boundary data, and V_1 assumes the least upper bound.

Equations (2.7) and (2.8) also apply to American options as we shall show with the following somewhat heuristic argument. Consider, for example, an American put for (2.1) with boundary data

$$V(s(t), t) = K - s(t)$$

$$V_S(s(t),t) = -1$$

$$V(X,t) = 0$$

initial data

$$s(0) = K$$

$$V(S,0) = 0, \qquad S \in [K, X]$$

and the continuation

$$V(S,t) = K - S, \qquad S \in [0, s(t)],$$

where $s(t)$ is the early exercise boundary and X is an arbitrary but fixed up and out barrier for the put chosen to replace the condition at infinity. We shall ignore the fact that the price of this put is convex and argue only on the basis of the nonlinear equations so that our considerations apply to portfolios with non-convex pay-off.

Let V_0 be the solution of the BSB equation (2.2)

$$\mathcal{L}(\sigma, r)V_0 = F_0(S, V_0, V_{0S}, V_{0SS})$$

(or equivalently, the solution of equation (2.7)), subject to the free boundary and initial conditions for an American put

$$V_0(s_0(t), t) = K - s_0(t),$$

$$V_{0S}(s_0(t), t) = -1$$

$$V_0(S, 0) = (K - S)^+$$

and the artificial barrier

$$V_0(X, t) = 0$$

for sufficiently large X. We shall assume that the solution $\{V_0(S,t), s_0(t)\}$ exists, is unique and smooth. It will be continued linearly over $(0, s_0(t))$ as

$$V_0(S,t) = K - S.$$

We intend to establish that again $V_0(S,t) \leq V(S,t)$.

Let $D = \{(S,t) : S \in (s_0(t), X),\ t \in (0,T]\}$. Then for all $S > \max\{s(t), s_0(t)\}$ we have for

$$e(S,t) = V_0(S,t) - V(S,t)$$

the inequality

$$\mathcal{L}(\sigma, r)e(S,t) = F_0(S, V_0, V_{0S}, V_{0SS}) \geq 0.$$

If $s_0(t) < s(t)$ then for all $S \in (s_0(t), s(t))$ we would have

$$\mathcal{L}(\sigma, r)e(S,t) = F_0(S, V_0, V_{0S}, V_{0SS}) + rK$$

because $\mathcal{L}(\sigma, r)V(S,t) = -Kr$ in the continuation region. At any rate, we know that

$$\mathcal{L}(\sigma, r)e(S,t) \geq 0$$

for all $S \subset D$ and $t > 0$ so that $e(S,t)$ cannot have positive maximum in D. Moreover, if $s_0(t) \geq s(t)$ then $e(s_0(t), t) \leq 0$ because V_0 takes on its intrinsic value and V lies above it to the right of $s(t)$. On the other hand, if $s_0(t) < s(t)$ then $e(s_0(t), t) = 0$ because V is in the continuation region. Since also $e(S,0) = e(X,0) = 0$ it follows that

$$e(S,t) \leq 0 \quad \text{everywhere in } D$$

so that

$$V_0(t) \leq V(S,t)$$

in D, and because of the linear continuation, in $(0,X) \times (0,T]$.

In fact, $s_0(t)$ will stay to the right of $s(t)$. If not then there is a time t^* such that $s_0(t^*) < s(t^*)$ and $e(s_0(t^*), t^*) = e_S(s_0(t^*), t^*) = 0$. The boundary data imply that

$$V_{0t}(s_0(t^*), t^*) = 0$$

while equation (2.5) shows that

$$\lim_{S \to s(t_0)+} \frac{1}{2} f_0(\sigma) S^2 V_{0SS} = g_0(r)K$$

so that $V_{0SS}(s(t^*), t^*) > 0$ which implies that $e_{SS}(s_0(t^*), t^*)$ is strictly positive. Hence $e(S, t^*)$ is strictly increasing with S at $s_0(t^*)$ and would become positive as we move into D. This contradicts that e has no positive maximum anywhere in D. We can conclude that

$$s(t) \le s_0(t) \quad \text{for } t \in (0, T).$$

A similar argument applied to equation (2.4) with the boundary data of an American put will yield

$$V(S,t) \le V_1(S,t) \quad \text{on } [0, X] \times [0, T].$$

To illustrate these comments we show below numerical results for a straddle similar to (2.11) where the down and out barrier is replaced by an early exercise boundary $s(t)$. The option price $V(S,t)$ is described by

$$\begin{aligned}
\mathcal{L}(\sigma, r)V(S,t) &= 0 \\
V(S,0) &= (100 - S)^+ + (S - 100)^+ \\
V(120, t) &= 0 \\
V(s(t), t) &= 100 - s(t) \\
V_S(s(t), t) &= -1.
\end{aligned} \tag{2.12}$$

Fig. 2.2a shows the envelopes V_0 and V_1 at $T = .1$ when

$$.1 = \sigma_0 \le \sigma \le \sigma_1 = .4$$

$$.04 = r_0 \le r \le r_1 = .06$$

which bound the Black Scholes solution $V(S,t)$ of (2.12) for $\sigma = .2$ and $r = .05$. Fig. 2.2b shows the early exercise boundaries as a function of time associated with V_0, V and V_1. Numerical values at expiry are

$$s_1(.1) = 74.63, \quad s(.1) = 88.19, \quad s_0(.1) = 94.94.$$

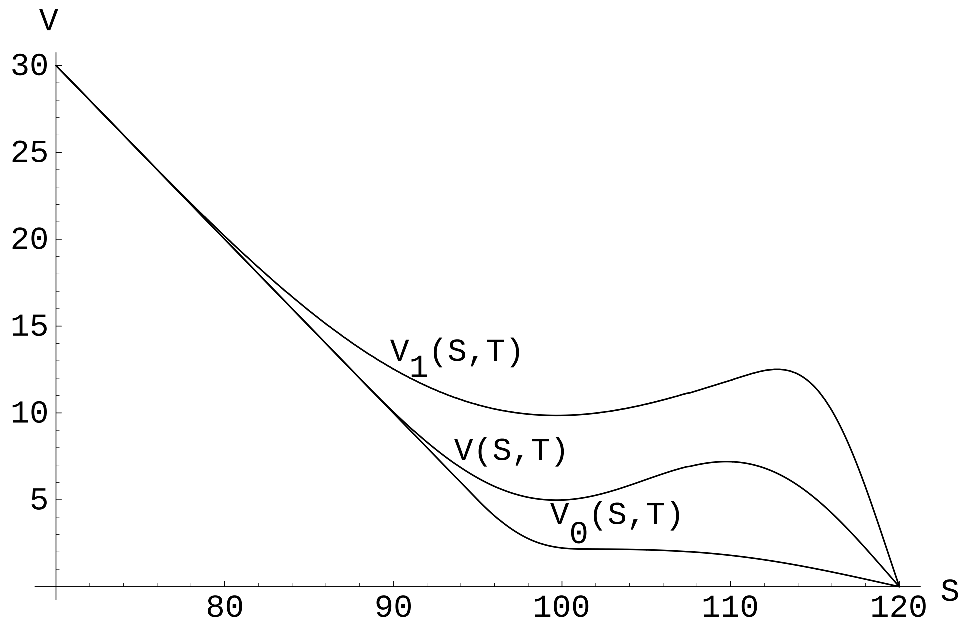

Fig. 2.2a. Envelopes V_0 and V_1 and Black Scholes solution V for the American straddle (2.12).

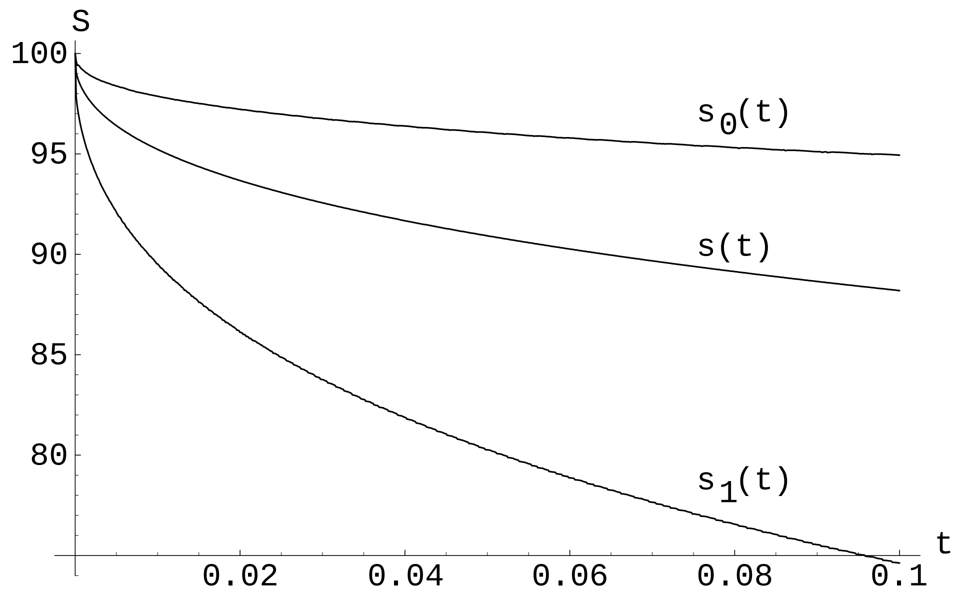

Fig. 2.2b. Early exercise boundaries for the American straddle (2.12).

3. Derivation of bounds for the delta

Since for hedging the delta V_S of the option is required we shall next examine how upper and lower bounds on V_S can be found. We look again at the up and out European call. To simplify the argument we shall assume that σ and r are either constant or functions of t only. If we now differentiate the Black Scholes equation (2.1) with respect to S we find that the delta $V_S \equiv \delta(S,t)$ satisfies the differential equation

$$M(\sigma, r)\delta \equiv \frac{1}{2}\,\sigma^2 S^2 \delta_{SS} + (\sigma^2 + r)S\delta_S - \delta_t = 0. \tag{3.1}$$

The initial condition for the up and out call is

$$\delta(S,0) = \begin{cases} 0 & S < K \\ 1 & S > K. \end{cases} \tag{3.2}$$

Since any solution of the Black Scholes equation for σ and r in their prescribed ranges has to fall between the envelopes V_0 and V_1 and since $V_0(0,t) = V_1(0,t)$ and $V_0(X,t) = V_1(X,t)$ it follows that for any admissible σ and r

$$V_{0S}(0,t) \leq \delta(0,t) \leq V_{1S}(0,t)$$

$$V_{1S}(X,t) \leq \delta(X,t) \leq V_{0S}(X,t).$$

Since V_0 and V_1 are computable their derivatives may be assumed known.

We are now ready to derive the model equation for a lower bound on $\delta(S,t)$. Consider the equation

$$\begin{aligned} M(\sigma, r)\delta_0 = {} & \frac{1}{2}\,(\sigma^2 - \sigma_0^2)S^2\delta^+_{0SS} + \frac{1}{2}\,(\sigma^2 - \sigma_1^2)S^2\delta^-_{0SS} \\ & + (\sigma^2 - \sigma_0^2)S\delta^+_{0S} + (\sigma^2 - \sigma_1^2)S\delta^-_{0S} \\ & + (r - r_0)S\delta^+_{0S} + (r - r_1)S\delta^-_{0S} \end{aligned} \tag{3.3}$$

subject to the boundary data

$$\begin{aligned} \delta_0(0,t) &= V_{0S}(0,t) \\ \delta_0(X,t) &= V_{1S}(X,t) \end{aligned} \tag{3.4}$$

and the initial data (3.2). By construction,

$$M(\sigma, r)\delta_0 \geq 0$$

and the right hand side is continuous. As before, we shall assume that discontinuous initial/boundary data are smoothed and that the resulting nonlinear problem has a solution $\delta_0(S,t)$ which is continuous on $[0,X] \times [0,T]$ and smooth on $(0,X) \times (0,T]$.

We now can apply maximum principle arguments. Let us set

$$e(S,t) = \delta_0(S,t) - \delta(S,t)$$

then

$$M(\sigma,r)e(S,t) \geq 0$$

$$e(0,t) \leq 0, \qquad e(S,0) = 0, \qquad e(X,t) \leq 0.$$

The maximum principle applies to the operator $M(\sigma,r)$ and assures that any solution of (3.3) must assume its maximum also at some point on the boundary $S = 0$, $t = 0$ or $S = X$. Since e is non-positive on the boundary we see that

$$e(S,t) \leq 0$$

or

$$\delta_0(S,t) \leq \delta(S,t).$$

To obtain an upper bound $\delta_1(x,t)$ we interchange the positive and negative parts in the right hand side of equation (3.3) and impose the boundary conditions

$$\delta_1(0,t) = V_{1S}(0,t)$$

$$\delta_1(X,t) = V_{0S}(X,t)$$

Then

$$M(\sigma,r)\delta_1(S,t) \leq 0.$$

If we now set

$$e(S,t) = \delta_1(S,t) - \delta(S,t)$$

then

$$M(\sigma,r)e(S,t) \leq 0$$

and

$$e(0,t) \geq 0, \qquad e(S,0) = 0, \qquad e(X,t) \geq 0.$$

Now the differential equation and the boundary conditions rule out an negative minimum for e. Hence

$$\delta_1(S,t) \geq \delta(S,t).$$

It is straightforward to find the two equivalent forms of equation (3.3). Since all terms involving σ and r cancel out in (3.3) we obtain the Black Scholes Barenblatt type equation

$$\frac{1}{2}\,\widehat{f}_0(\sigma)^2 S^2 \delta_{0SS} + [\widehat{k}_0(\sigma)^2 + \widehat{g}_0(r)] S\delta_{0S} - \delta_{0t} = 0 \tag{3.5}$$

where

$$\widehat{f}_0(\sigma) = \begin{cases} \sigma_0 & \text{if } \delta_{0SS} \geq 0 \\ \sigma_1 & \text{if } \delta_{0SS} < 0 \end{cases}$$

$$\widehat{k}_0(\sigma) = \begin{cases} \sigma_0 & \text{if } \delta_{0S} \geq 0 \\ \sigma_1 & \text{if } \delta_{0S} < 0 \end{cases}$$

and

$$\widehat{r}_0(r) = \begin{cases} r_0 & \text{if } \delta_{0S} \geq 0 \\ r_1 & \text{if } \delta_{0S} < 0. \end{cases}$$

Alternatively, writing $\delta^+_{0SS} = (\delta_{0SS} + |\delta_{0SS}|)/2$ etc. yields the symmetric form

$$\begin{aligned} M(\bar{\sigma},\bar{r})\delta_0 &\equiv \frac{1}{2}\,\bar{\sigma}^2 S^2 \delta_{0SS} + \left[\bar{\sigma}^2 + \bar{r}\right] S\delta_{0S} - \delta_{0t} \\ &= \frac{1}{2}\left(\frac{\sigma_1^2 - \sigma_0^2}{2}\right) S^2 |\delta_{0SS}| + \left[\left(\frac{\sigma_1^2 - \sigma_0^2}{2}\right) + \left(\frac{r_1 - r_0}{2}\right) S\right] |\delta_{0S}|. \end{aligned} \tag{3.6}$$

If the right hand side of equation (3.6) is denoted by $\bar{G}(S,\delta_0,\delta_{0S},\delta_{0SS})$ then the equation for δ_1 is

$$M(\bar{\sigma},\bar{r})\delta_1 = -\bar{G}(S,\delta_1,\delta_{1S},S_{1SS}).$$

But note that δ_0 and δ_1 assume different boundary conditions.

As a numerical example we show in Fig. 3.1 bounds for the delta as well as the delta of a Black Scholes solution at $t = T = .1$ for an up and out call described by

$$\mathcal{L}(\sigma,r)V(S,t) = 0 \tag{3.7}$$

$$V(S,0) = (S-100)^+$$

$$V(0,t) = V(120,t) = 0.$$

The bounds correspond to a range of

$$.2 = \sigma_0 \leq \sigma \leq \sigma_1 = .4$$

$$.04 = r_0 \leq r \leq r_1 = .06$$

while the Black Scholes δ corresponds to $\sigma = .3$ and $r = .05$.

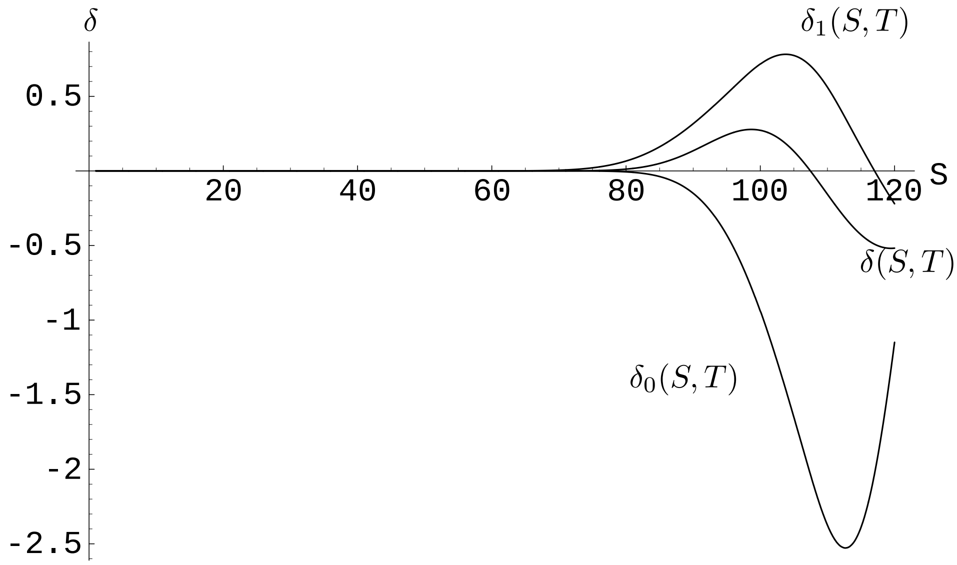

Fig. 3.1. Delta bounds δ_0, δ_1 and δ for the up and out European call (3.7).

We point out that the bounding curves δ_0 and δ_1 are not envelopes because they correspond to discontinuous σ and r for which (3.1) does not hold.

The exposition of Sections 2 and 3 shows that the derivation of the BSB equation is automatic and readily extended. For example, if the function V satisfies

$$\mathcal{L}(\sigma, r)V = f(\sigma, r, S, t)$$

then the BSB equation corresponding to (2.7), (2.8) are

$$\mathcal{L}(\sigma, r)V_0 = F(S, V_0, V_{0S}, |V_{0SS}|) + \max_{\sigma,r} f(\sigma, r, S, t) \tag{3.8}$$

$$\mathcal{L}(\sigma, r)V_1 = -F(S, V_1, V_{1S}, |V_{1SS}|) + \min_{\sigma,r} f(\sigma, r, S, t). \tag{3.9}$$

Similarly, bounds for the solution of the Black Scholes equation for a basket of options follow immediately from the maximum principle for higher dimensional diffusion equations.

For example, suppose we have an option on two underlying assets. Then its price is given by

$$\mathcal{L}(\rho)V \equiv \frac{1}{2}\sigma_1^2 S_1^2 V_{S_1S_1} + \rho\sigma_1\sigma_2 S_1 S_2 V_{S_1S_2}$$

$$\frac{1}{2}\,\sigma_2^2 S_2^2 V_{S_2S_2} + rS_1 V_{S_1} + rS_2 V_{S_2} - rV - V_t = 0.$$

For simplicity, let us assume that only the correlation factor ρ is unknown and bounded, i.e.,

$$-1 \leq \rho_0 \leq \rho \leq \rho_1 \leq 1.$$

Then the equation analogous to (2.2) is

$$\mathcal{L}(\rho)V = \sigma_1\sigma_2 S_1 S_2 \left[(\rho - \rho_0)\, V^+_{S_1S_2} + (\rho - \rho_1) V^-_{S_1S_2}\right].$$

Expressions corresponding to (2.5) and (2.7) are readily found. In particular, the expression analogous to (2.7) becomes

$$\mathcal{L}\left(\frac{\rho_0 + \rho_1}{2}\right) V_0 = -\sigma_1\sigma_2 S_1 S_2 \left(\frac{\rho_1 - \rho_0}{2}\right) |V_{0S_1S_2}|. \tag{3.10}$$

The initial and boundary conditions for V_0 are the same as those for V. We do not know whether the nonlinear equation (3.10) remains solvable with a simple fixed point iteration.

To conclude this section let us turn our arguments around and consider for fixed σ the equation

$$\mathcal{L}(\sigma, r)V_0 = \frac{1}{2}\,\sigma^2 Le(\sigma) S^2 |V_{0SS}| \tag{3.11}$$

which arises when transaction costs are modeled in a Black Scholes setting. Here $Le(\sigma) \geq 0$ is the so-called Leland number given as

$$Le(\sigma) = \frac{K}{\sigma}\,, \tag{3.12}$$

where K is a constant depending on the level of transaction cost and the time interval between rehedging of the portfolio (see, e.g. [12]).

A comparison with equation (2.7) shows that (3.11) can be identified with the Black Scholes Barenblatt equation for the lower envelope of the solution $V(S, t)$ of the linear Black Scholes equation

$$\mathcal{L}(\widehat{\sigma}, r)V = 0 \tag{3.13}$$

where $\widehat{\sigma}$ is a fixed volatility function subject to lower and upper bounds

$$\widehat{\sigma}_0(\sigma) \leq \widehat{\sigma} \leq \widehat{\sigma}_1(\sigma)$$

computed from

$$\frac{\widehat{\sigma}_1^2 + \widehat{\sigma}_0^2}{2} = \sigma^2$$

$$\frac{\widehat{\sigma}_1^2 - \widehat{\sigma}_0^2}{2} = \sigma^2 Le(\sigma).$$

Adding and subtracting these two equations yield

$$\widehat{\sigma}_0(\sigma) = \sigma\sqrt{1 - Le(\sigma)}$$

$$\widehat{\sigma}_1(\sigma) = \sigma\sqrt{1 + Le(\sigma)}.$$

If we now assume that σ itself is uncertain but bounded according to

$$\sigma_0 \leq \sigma \leq \sigma_1$$

then we see that the lower envelope for (3.13) corresponding to

$$\begin{aligned}\widehat{\sigma}_0(\sigma_0) &= \min_{\sigma} \widehat{\sigma}_0(\sigma) = \sigma_0\sqrt{1 - \frac{K}{\sigma_0}} \\ \widehat{\sigma}_1(\sigma_1) &= \max_{\sigma} \widehat{\sigma}_1(\sigma) = \sigma_1\sqrt{1 + \frac{K}{\sigma_1}}\end{aligned} \tag{3.14}$$

is a lower bound on the solution of (3.12) for all $\sigma \in [\sigma_0, \sigma_1]$ provided $K/\sigma_0 \leq 1$.

The Black Scholes Barenblatt equation for (3.13) corresponding to the lower and upper bounds of (3.14) is the equation derived directly in [1] under the above condition on the Leland number by balancing the return on a portfolio.

4. Narrowing the envelope with static hedging

It has been observed that unless the bounds on the volatility and interest rates are very tight the envelope for the option price may be too broad to be financially significant. But if related instruments already are traded then it is suggested (see, e.g., [3, 12]) to price a suitable portfolio of options in order to narrow the envelope. The option is said to be statically hedged with this portfolio. Assuming only the validity of the Black Scholes equation, and relying on the maximum principle, we shall describe static hedging in the context of PDEs with uncertain volatility, clarify the underlying assumptions, define the optimal static hedge and discuss analytic and computational approximations to it.

We begin by considering static hedging of a European option with other European options with the same time to expiry. To be specific, suppose we wish to price a European option today with strike price K and expiration T. We again set $t = T - \tau$ where τ is calendar time. If we denote the option value by $V(S,t)$ then it is assumed to satisfy the Black Scholes equation

$$\mathcal{L}(\sigma, r)V(S,t) = 0 \tag{4.1}$$

subject to the initial condition and boundary conditions which characterize the option. As before, σ and r are considered uncertain but bounded above and below. We can estimate the dependence of $V(S,t)$ on σ and r today from

$$V_0(S(T),T) \leq V(S(T),T) \leq V_1(S(T),T)$$

where V_0 and V_1 satisfy (2.7) and (2.8). These bounds are sharp and cannot be improved because V_0 and V_1 are the Black Scholes prices when σ and r coincide with the lower and upper bounds according to equations (2.5), (2.6).

Let us now assume that N other European options on the same asset with the same expiration and various pay-offs at $t = 0$ are freely traded. We shall denote their quoted prices today by $\{W_i(S(T),T)\}_{i=1}^N$. As usual, we also know their values at expiration. For the time being we also suppose that V and each W_i are defined on a common interval $[S_0, S_1] \subset [0, \infty)$ with known boundary data at S_0 and S_1.

We make the essential assumption that the (unknown) option prices $\{W_i(S,t)\}_{i=1}^N$ as functions of S and t reflect the market and thus follow the Black Scholes equation (4.1) for the same uncertain volatility and interest functions as the option $V(S,t)$ which we wish to bound. However, the only data we have for each $W_i(S,t)$ are its value at expiration $t = 0$, the boundary data, and the quoted price today, i.e. the restriction $W_i(S(T),T)$.

Consider now the portfolio

$$\pi(S,t,\vec{c}) = V(S,t) - \sum_{i=1}^{M} c_i W_i(S,t) \tag{4.2}$$

where $\vec{c}$ denotes the set $\{c_1, \ldots, c_M\}$ of real numbers which are yet to be determined. Once $\vec{c}$ has been chosen the components $\{c_i W_i\}$ of the portfolio remain unchanged (i.e. static) for the life of the portfolio. $\sum_{i=1}^{M} c_i W_i(S,t)$ is called a static hedge of $V(S,t)$.

It follows from the linearity of the Black Scholes operator $\mathcal{L}(\sigma, r)$ that

$$\mathcal{L}(\sigma, r)\pi(S,t,\vec{c}) = 0.$$

The initial condition is given analytically as

$$\pi(S,0,\vec{c}) = V(S,0) - \sum_{i=1}^{M} c_i W_i(S,0).$$

In addition $\pi(S,t,\vec{c})$ is subject to known boundary data at S_0 and S_1. For any given choice of numbers $\{c_i\}$ the portfolio $\pi(S,t,\vec{c})$ can be bounded below with the BSB equation (2.7) as described in Section 2.

Moreover, the expression

$$V(S(T),T) = \pi(S(T),T,\vec{c}) + \sum_{i=1}^{M} c_i W_i(S(T),T)$$

implies that for given $\{c_i\}$ a lower bound π_0 for $\pi(S(T),T,\vec{c})$ gives a lower bound on $V(S(T),T)$ valid for all σ and r in their specified ranges.

Similarly, an upper bound π_1 for a portfolio

$$\pi(S,t,\vec{d}) = V(S,t) - \sum_{j=1}^{N} d_j W_j(S,t) \tag{4.3}$$

yields an upper bound on $V(S,t)$. Note that the lower and upper bounds are logically independent of each other so that portfolios (4.2) and (4.3) need not have the same number and kind of options. In particular, differences in bid-ask prices can be accounted for.

We also remark that in view of (2.10) lower and upper bounds on the portfolios $-\pi(S,t,\vec{c})$ and $-\pi(S,t,\vec{d})$ yield the same upper and lower bounds on $V(S(T),T)$.

The critical issue in static hedging is the determination of the coefficients $\{c_i\}$ and $\{d_i\}$. If we wish to sell $V(S,t)$ then the choice of $\{c_i\}$ which minimizes the function

$$E_u(\vec{c}) = \pi_1(S(T),T,\vec{c}) + \sum_{i=1}^{M} c_i W_i(S(T),T) \tag{4.4}$$

leads to the least upper bound on $V(S(T),T)$ which guarantees that the portfolio is not underpriced for any volatility and interest rate function in the prescribed range. Similarly, if we plan to buy $V(S,t)$ then the solution $\{d_i\}$ which maximizes

$$E_e(\vec{d}) = \pi_0(S(T),T,\vec{d}) + \sum_{j=1}^{N} d_j W_j(S(T),T) \tag{4.5}$$

gives the greatest lower bound on $V(S(T),T)$ below which the portfolio is not overpriced for any volatility and interest rate in the given range.

The solution $\{c_i\}$ which minimizes (4.4) defines the optimal hedge for the seller of the option V. The solution $\{d_i\}$ which maximizes (4.5) defines the optimal hedge for the buyer of V. We set

$$\widehat{V}_1(S(T),T) = \min_{c_1,\ldots,c_M} E_u(\vec{c})$$

$$\widehat{V}_0(S(T),T) = \max_{d_1,\ldots,d_N} E_e(\vec{d})$$

and note that

$$V_0(S(T),T) \le \widehat{V}_1(S(T),t) \le E_u(\vec{0}) = V_1(S(T),T)$$

$$V_0(S(T),T) = E_\ell(\vec{0}) \le \widehat{V}_0(S(T),T) \le V_1(S(T),T)$$

where V_0 and V_1 are the BSB solutions at $t=T$.

We have no proof that the optimal hedges exist because the mappings of $\vec{c} \to E_u(\vec{c})$ and $\vec{d} \to E_\ell(\vec{d})$ stand for the relationship between point values of solutions of the BSB equation and its initial and boundary conditions. Even for linear diffusion problems such relationship is not obvious. In the examples considered in Section 5 we hedge with only one or two options. There a minimization can be carried out numerically with a simple search.

The arguments remain unchanged if the options $\{W_i\}$ expire at different times. For ease of illustration we shall hedge with only one traded option $W(S,t)$ with expiry T_1. If $T_1 < T$ then the pay-off $W(S, T-T_1)$ is known analytically and the portfolio $\pi(S,t,c)$ has the envelopes

$$\pi_0(S, T-T_1, c) = V_0(S, T-T_1) - cW(S, T-T_1)$$

$$\pi_1(S, T-T_1, c) = V_1(S, T-T_1) - cW(S, T-T_1)$$

where V_0 and V_1 are found over $(0, T-T_1)$ from (2.7), (2.8). Note that as a function of t the volatility $\sigma(S,t)$ and interest rate $r(S,t)$ yielding the envelopes over $(0, T-T_1]$ and $(T-T_1, T]$ will be discontinuous because V_0 and π_0 will in general have different convexity at $t = T - T_1$. Similarly, if $T_1 > T$ then the initial conditions are

$$\pi_0(S, 0, c) = V(S, 0) - \max\{cW_0(S, 0), cW_1(S, 0)\}$$

$$\pi_1(S, 0, c) = V(S, 0) - \min\{cW_0(S, 0), cW_1(S, 0)\}$$

where W_0 and W_1 are found over $(T-T_1, 0)$ from (2.7), (2.8). The envelope equations for W and π determine $\sigma(S,t)$ and $r(S,t)$ for $t \in (T-T_1, 0]$ and $(0,T]$, resp. This construction is easily generalized to static hedging with N options with different lifetimes. As long as all options are defined on the same interval $[S_0, S_1]$ one simply adds a new time slice for each new W.

Envelopes of portfolios can generally not be computed exactly if the static hedge is carried out with options defined on different intervals. For example, suppose we wish to hedge a barrier option V described by

$$\mathcal{L}(\sigma, r)V = 0$$

$$V(0,t) = 0, \qquad V(S_1, t) = f(t)$$

with an up and out barrier call option W described by

$$\mathcal{L}(\sigma, r)W = 0$$

$$W(0,t) = W(X,t) = 0$$

where $S_1 < X_1$. For simplicity we shall assume expiry T for both options. Consider the portfolio $\pi(S,t,c) = V(S,t) - cW(S,t)$ where c is given real number. To find its envelopes over $[0, S_1]$ we need the boundary value $\pi(S_1,t,c) = V(S_1,t) - cW(S_1,t)$. But $W(S_1,t)$ is not computable because σ and r are not known with certainty. To find a lower bound on π let $\phi(S,t,c)$ be the solution of

$$\mathcal{L}(\sigma, r)\widehat{\phi} = 0$$

$$\phi(S_0,t) = 0, \qquad \phi(S_1,t,c) = V(S_1,t) - \max cW(S_1,t) \tag{4.6}$$

$$\phi(S,0,c) = V(S,0) - cW(S,0)$$

where the maximum is taken over all admissible σ and r. We know that for any real c

$$\max cW(S_1,t) = \max\{cW_0(S_1,t), cW_1(S,t)\}$$

where W_0 and W_1 are the lower and upper envelopes for $W(S,t)$. The maximum principle assures that

$$\phi(S,t,c) \le \pi(S,t,c)$$

for all admissible σ and r. It follows that the lower envelope $\phi_0(S,t,c)$ for problem (4.6) yields a lower bound on the portfolio $\pi(S,t,c)$. This lower bound for $\phi(S,t,c)$ cannot be attained, i.e. is not an envelope, whenever W_{SS} and ϕ_{SS} or $SW_S - W$ and $S\phi_S - \phi$ have different algebraic signs anywhere in $[0, S_1]$ because in this case the envelopes for W and ϕ require different volatility or interest rate functions.

Suppose in the above discussion W itself is a portfolio consisting of N options so that

$$W(S,t) = \sum_{n=1}^{N} c_i W_i(S,t)$$

with $\{W_i(S,t)\}$ defined on intervals $\{[S_0, X_i]\}$ such that $S_1 < X_1 < \cdots < X_N$. Then the boundary condition for $\phi(S,t,\vec{c})$ is

$$\phi(S_1,t,\vec{c}) = V(S_1,t) - \max\left\{\sum_{n=1}^{N} c_i W_i(S,t)\right\}.$$

An upper bound on $W(S_1,t)$ can be found recursively by considering increasingly larger subportfolios. It is possible that the simpler approximation

$$\phi(S_1,t,\vec{c}) = V(S_1,t) - \sum_{n=1}^{N} \max c_i W_i(S_1,t)$$

will be sufficiently accurate because solutions to the Black Scholes equation at interior points like $S(T)$ are usually sensitive to initial but not boundary data.

If hedging is done with N options with two-sided barriers then the computational effort to approximate the boundary condition for the portfolio $\pi(S,t,c)$ increases enormously, particularly when the intervals of the barrier options are not nested. This issue is addressed in [2] where the Black Scholes Barenblatt equations in the form of (2.5), (2.6) are solved with an explicit finite difference method.

Defining a static hedge for American options is more complicated because there may be early exercise. To illustrate the PDE approach in this case consider the specific scenario where an American put $V(S,t)$ with strike K and expiration T is to be hedged with an American put $W(S,t)$ with the same time to expiry but a different strike price K_1. We shall denote the early exercise boundary of V and W by $s(t)$ and $s_1(t)$, resp. Then V and W are solutions of

$$\mathcal{L}(\sigma,r)V = -rKH(s(t)-S), \qquad 0 < S < \infty, \quad t > 0$$

$$V(S,0) = (K-S)^+$$

$$V(0,t) = K, \qquad \lim_{S\to\infty} V(S,t) = 0$$

and

$$\mathcal{L}(\sigma,r)W = -rK_1H(s_1(t)-S), \qquad 0 < S < \infty, \quad t > 0$$

$$W(S,0) = (K_1-S)^+$$

$$W(0,t) = K_1, \qquad \lim_{S\to\infty} W(S,t) = 0$$

where $H(x)$ is the Heaviside function. We look again at the portfolio

$$\pi(S,t,c) = V(S,t) - cW(S,t).$$

Then

$$\mathcal{L}(\sigma, r)\pi = r[cK_1 H(s_1(t) - S) - KH(s(t) - S)]. \tag{4.7}$$

We also know that

$$\begin{aligned} \pi(S, 0, c) &= (K - S)^+ - c(K - S)^+ \\ \pi(0, t, c) &= K - cK_1 \\ \lim_{S \to \infty} \pi(S, t, c) &= 0. \end{aligned} \tag{4.8}$$

We need lower and upper bounds on $\pi(S(T), T, c)$ which hold uniformly with respect to σ and r. They are not computable from (3.8), (3.9) because we do not know $s(t)$ and $s_1(t)$ in the source term of (4.7). However, we do know a priori for an American put that

$$s(t) \leq K; \qquad s_1(t) \leq K_1.$$

Let $\phi(S, t, c)$ be the solution of

$$\mathcal{L}(\sigma, r)\phi = \begin{cases} rcK_1 H(K_1 - S), & c \geq 0 \\ 0, & c < 0 \end{cases} \tag{4.9}$$

subject to (4.8) then

$$\mathcal{L}(\sigma, r)(\phi - \pi) \geq 0.$$

The maximum principle guarantees that

$$\phi(S, t, c) \leq \pi(S, t, c)$$

so that

$$[\phi_0(S(T), T, c) + cW(S(T), T)] \leq V(S(T), T).$$

$\phi_0(S, t, c)$ is found from (3.8) with $f(S, t) = \max[0, cr_1 K_1 H(K_1 - S)]$. An approximation to the optimal static hedge is given by the constant c which yields the greatest lower bound on $V(S(T), T)$.

An upper bound on $V(S(T), T)$ is found in terms of $\psi_1(S, t, d)$ where ψ_1 is the solution of the BSB equation (3.9) associated with

$$\mathcal{L}(\sigma, r)\psi = -rKH(K - S) + \min[0, rcK_1 H(K_1 - s)]$$

and the initial/boundary conditions (4.7) (with $d \to c$). For ease of exposition we shall also use the notation

$$E_u(\vec{c}) = \psi_1(S(T), T, \vec{c}) + \sum_{i=1}^{M} c_i W_i(S(T), T)$$

and

$$E_1(\vec{d}) = \phi_0(S(T), T, \vec{c}) + \sum_{j=1}^{M} c_j W_j(S(T), T)$$

when $\pi(S, t, \vec{c})$ is not computable and has to be bounded by approximating functions ϕ and ψ.

5. Numerical examples

The following numerical examples are meant to illustrate the PDE approach to static hedging, alert the reader to numerical difficulties in determining the optimal static hedge, and to indicate that the BSB equations are usually as simple to solve numerically as the Black Scholes equation itself. A detailed financial study comparing approximate Barenblatt envelopes with Monte Carlo simulations of option prices for a stochastic volatility may be found in [3] and an extension of the study for hedging with transaction costs is given in [1].

All our examples involve hedging with only one or two traded options. This allows the minimization of $E_u(c)$ and $-E_\ell(d)$ with a simple search on a grid in parameter space. For hedging with more than two options a simple search would no longer be feasible but require a more systematic approach such as a Nelder Mead type search on simplices [6] (also outlined in [12]).

To illustrate static hedging we consider first the case of pricing a plain European call with $T = .5$. $K = S = 100$ and $r = .05$ already discussed in [12]. For the volatility we impose the bounds

$$.2 \leq \sigma(S, t) \leq .3.$$

Since the solution for a call is convex the BSB equations (2.5), (2.6) reduce to the standard Black Scholes equations

$$\mathcal{L}(.3, .5)V_0 = 0$$

$$\mathcal{L}(.2,.5)V_1 = 0$$

with the initial and boundary condition for a plain vanilla call. The Black Scholes formula yields the values

$$6.89 = V_0(100,.5) < V(100,.5) < V_1(100,.5) = 9.64.$$

As pointed out in [12] the spread $V_1(100,.5) - V_0(100,.5) = 2.75$ is too large for pricing the call. The buyer would not pay V_1 and the seller would not accept V_0.

We shall look at three different scenarios of static hedging for the above call to see how V can be priced more competitively. The first example has an analytic solution and is chosen primarily to characterize the optimal hedge and to verify that the numerical method reproduces an analytic solution.

Case 1: We assume that a European put $W(S,t)$ with $K = 100$, $T = .5$ and $r = .05$ is priced at $W(100,.5) = 5.791$, which corresponds to an implied volatility of $\sigma_{\text{imp}} = .25$. For the static hedge we consider the portfolio

$$\pi(S,t,c) = V(S,t) - cW(S,t).$$

It satisfies the Black Scholes equation

$$\mathcal{L}(\sigma,r)\pi = 0 \tag{5.1}$$

and the initial and boundary conditions

$$\pi(S,0,c) = (S-K)^+ - c(K-S)^+ \tag{5.2}$$

$$\lim_{S\to\infty} \pi(S,t,c) = \lim_{S\to\infty} (S - Ke^{-rt}) \tag{5.3}$$

$$\pi(0,t,c) = -cKe^{-rt}. \tag{5.4}$$

For $c \geq 1$ the initial condition $\pi(S,0,c)$ is convex downward and hence $\pi_0(S,t,c)$ is the Black Scholes solution for $\sigma = .3$. Similarly, for $c < 1$ π_0 is the Black Scholes solution for $\sigma = .2$. Then we can write

$$E_\ell(c) = \begin{cases} V(S(T),T) - cW_{.2}(S(T),T) + cW_{.25}(S(T),T), & c < 1 \\ V(S(T),T) - cW_{.3}(S(T),T) + cW_{.25}(S(T),T), & c \geq 1 \end{cases}$$

where the subscripts of W denote the applicable volatility. We can rewrite $E_\ell(c)$ with the put call parity relation

$$V(S,t) - W(S,t) = S - Ke^{-rt}$$

as

$$E_\ell(c) = S(T) - Ke^{-rT} + \begin{cases} W_{.2} + c(W_{.25} - W_{.2}), & c < 1 \\ W_{.3} + c(W_{.25} - W_{.3}), & c \geq 1 \end{cases}$$

An analogous derivation yields

$$E_u(c) = S(T) - Ke^{-rT} + \begin{cases} W_{.3} + c(W_{.25} - W_{.3}), & c < 1 \\ W_{.2} + c(W_{.25} - W_{.2}), & c \geq 1 \end{cases}$$

Hence $E_\ell(c)$ and $E_u(c)$ are piecewise linear and yield the optimum hedge at $c = 1$. We observe that the non-differentiability of $E_\ell(c)$ and $E_u(c)$ generally will require the tools of non-smooth optimization when solving for the optimal hedge.

In order to solve the problem numerically it is generally necessary to enforce the boundary condition given at infinity at a finite barrier $S = X$. In this and all subsequent examples we shall set

$$X = 3K.$$

In all cases the influence of changes in X on the computed answers is negligible.

When we solve the equations (2.7), (2.8) corresponding to (5.1)–(5.4) with c increasing in steps of $\Delta c = .1$ from -0.5 to 1.5 we find the values for $E_\ell(c)$ and $E_u(c)$ shown in Fig. 5.1.

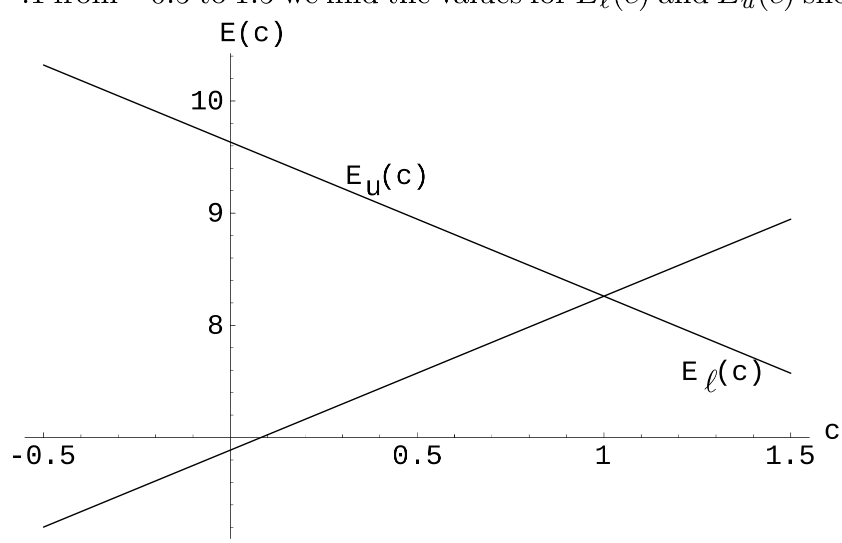

Fig. 5.1. Upper bound $E_u(c)$ and lower bound $E_\ell(c)$ for (5.1)–(5.4)

Tabulated values show that

$$\widehat{V}_1(S(T),T) = \max_c E_\ell(c) = \min_c E_u(c) = \widehat{V}_0(S(T),T) = 8.26$$

at $c = 1$ (as well as $E_u(0) = 6.89$ and $E_1(0) = 9.64$). Hence the numerical method finds the optimal hedge for the upper and lower bounds, a zero spread and the value of the call predicted by put-call parity. Note that the numerical method does not know about put-call parity or convexity of the solution but instead solves the fully nonlinear problems (2.7), (2.8).

Case 2: Let us assume that the value of $W(100,.5) = 5.79$ is given for an American put instead. This case requires a slight change of the argument given above for hedging American options. The portfolio remains subject to the initial condition (5.2) and the boundary condition (5.3). The boundary condition (5.4) is replaced by

$$\pi(S,t,c_1) = -cK.$$

$\pi(S,t,c)$ satisfies the inhomogeneous Black Scholes equation

$$\mathcal{L}(\sigma,r)\pi = rcKH(s(t)-S)$$

where $s(t)$ is the early exercise boundary of the put $W(S,t)$.

We know a priori that $s(t) \leq K$, but beyond that $s(t)$ is not available from market data. This complication forces us into an approximation of the optimum static hedge. Let $\phi(S,t,c)$ and $\psi(S,t,d)$ be solutions of the following problems

$$\mathcal{L}(\sigma,r)\phi = \max\{0, rdKH(K-S)\} \tag{5.5}$$

$$\phi(S,0,c) = \pi(S,0,c)$$

$$\phi(0,t,c) = \pi(0,t,c)$$

$$\phi(X,t,c) = \pi(X,t,c)$$

and

$$\mathcal{L}(\sigma, r)\psi = 0 \tag{5.6}$$
$$\psi(S, 0, d) = \pi(S, 0, d)$$
$$\psi(0, t, d) = \pi(S, 0, d)$$
$$\psi(X, t, c) = \pi(X, t, c).$$

It follows that for all d

$$\mathcal{L}(\sigma, r)(\phi - \pi) \leq 0$$

$$\mathcal{L}(\sigma, r)(\psi - \pi) \geq 0.$$

The maximum principle and the boundary data assure that for all c we have the inequalities

$$\phi(S(T), T, c) \leq \pi(S(T), T, c)$$

$$\pi(S(T), T, d) \leq \psi(S(T), T, d).$$

Hence a lower bound on ϕ and an upper bound on ψ provide lower and upper bounds on $V(S(T), T)$.

A lower bound ϕ_0 is found by solving the BSB equation (3.8) associated with the inhomogeneous Black Scholes equation (5.5). An upper bound ψ_1 is found by solving the BSB equation (2.8) associated with the homogeneous Black Scholes equation (5.6). The results for $E_1(c)$ and $E_u(d)$ are shown in Fig. 5.2 for c increasing from $-.5$ to 1.5 in steps of $\Delta c = .1$. Note that E_ℓ and E_u are now defined in terms of ϕ_0 and ψ_1. The curve for $E_u(c)$ is practically identical to $E_u(c)$ in Fig. 5.1 and yields $\widehat{V}_1(S(T), T) = 8.26$. This is not surprising since (5.6) almost describes a European call (except for the time discounted boundary condition). The maximum for $E_\ell(c)$ occurs at (i.e. around) $c = .7$ and yields

$$\widehat{V}_0(S(T), T) = 7.03$$

which is a modest improvement of $V_0(S(T), T) = 6.89$.

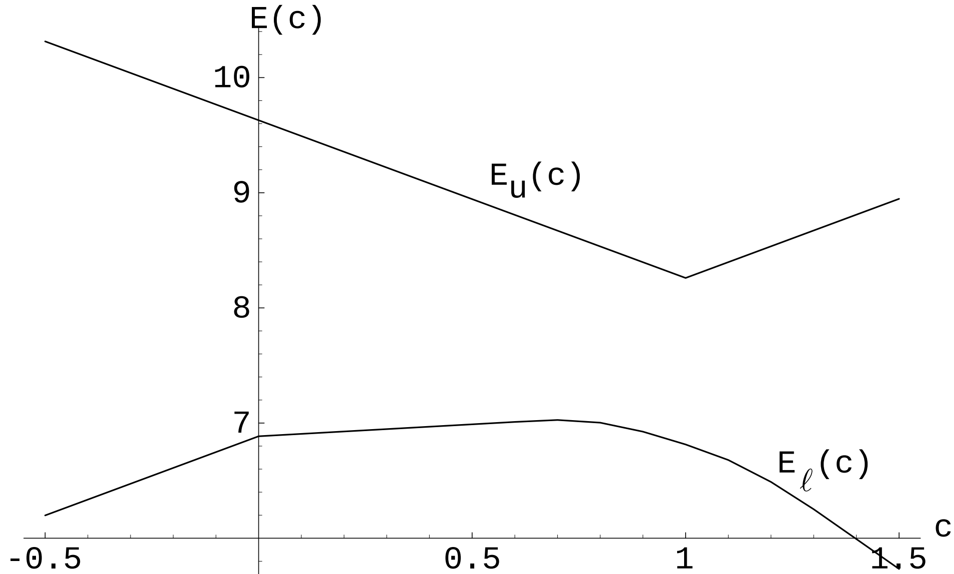

Fig. 5.2. $E_u(c)$ and $E_\ell(c)$ for hedging a European call with an American put

Case 3: We use the data of [12] and assume that two European call options W_1 and W_2 with strikes of $K_1 = 90$ and $K_2 = 110$ are traded at option prices of $W_1(100, .5) = 14.42$ and $W_2(100, .5) = 4.22$, respectively (both of which correspond to an implied volatility of $\sigma_{\text{imp}} = .25$). We now consider the portfolio

$$\pi(S, t, c) = V(S, t) - c_1 W(S, t) - c_2 W(S, t). \tag{5.7}$$

Then we need to solve

$$\mathcal{L}(\sigma, r)\pi = 0$$

subject to the initial condition

$$\pi(S, 0, c) = (S - 100)^+ - c_1(S - 90)^+ - c_2(S - 110)^+$$

and the boundary conditions

$$\pi(0, t, c) = 0$$

$$\pi(X, t, c) = (X - 100e^{-rt}) - c_1(X - 90e^{-rt}) - c_2(X - 110e^{-rt}).$$

We note that the boundary value $\pi(X, t, c)$ grows linearly with X unless $c_1 + c_2 = 1$. In fact, for $c_1 = c_2 = .5$ we obtain $\pi(X, t, c) = 0$ which would appear to be an attractive choice since it allows π to decay to 0 as $t \to \infty$. However, there is no mathematical method to predict quantitatively the influence of the boundary and initial conditions on the point value $\pi(100, .5, c)$ and as we shall see $c_1 = c_2 = .5$ does not yield the optimal portfolio.

We shall compute $E_\ell(c)$ and $E_u(c)$ as c_1 and c_2 increase from 0 to 1 in steps of $\Delta c = .1$. Fig. 5.3 shows the surface $E_u(c_1, c_2)$. The table of values for $E_\ell(c)$ and $E_u(c)$ yields

$$\widehat{V}_0(S(T), T) = 7.760 \text{ at } (c_1, c_2) = (.6, .6)$$

and

$$\widehat{V}_1(S(T), T) = 8.667 \text{ at } (c_1, c_2) = (.7, .5).$$

We remark that the optimum hedge was computed for a barrier at $X = 300$. It changes by less than one cent if $X = 1200$, which reflects the observation that the boundary condition at X has little influence on the solution of the BSB equation at $S = 100$.

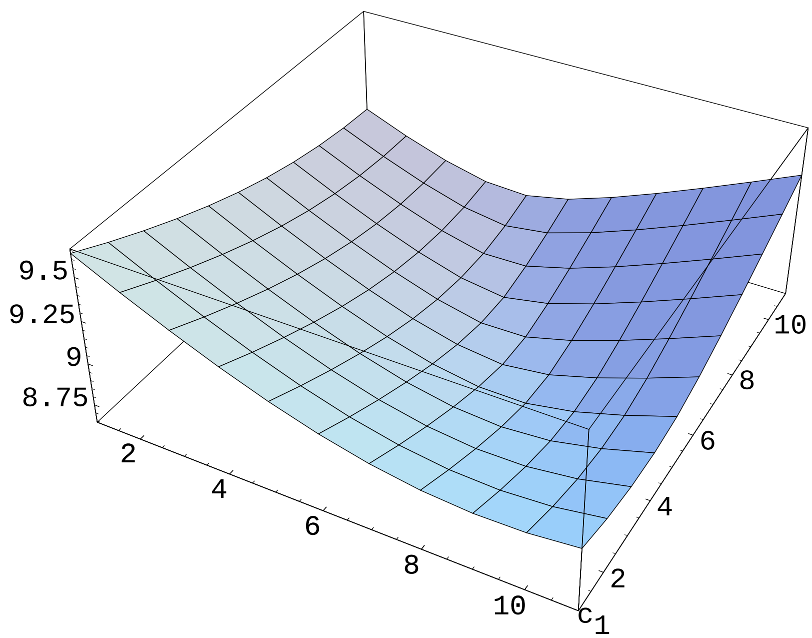

Fig 5.3. Surface $E_u(c_1, c_2)$ over the unit square for portfolio (5.7)

We note that the computed spread of the unhedged call of \$2.745 (for $c_1 = c_2 = 0$) coincides with the analytic solution obtained with the Black Scholes formula as reported in [12]. However, from our table of values we find that

$$E_u(.5, .5) - E_\ell(.5, .5) = 1.038.$$

This spread differs from the value of 1.38 reported in [12].

In our last numerical example we would like to hedge statically an up and out European call with a plain vanilla European call. The up and out call is described by

$$T = .5, \qquad K = S(T) = 100$$

$$\mathcal{L}(\sigma, r)V = 0$$

$$V(0,t) = V(120,t) = 0$$

$$V(S,0) = (S-100)^+.$$

We shall assume that

$$.2 = \sigma_0 \le \sigma \le \sigma_1 = .4$$

$$.03 = r_0 \le r \le .06.$$

Available for static hedging is a plain vanilla call $W(S,t)$ with $K = 90$ and traded at

$$W(100,0) = 15.486$$

(which corresponds to an implied volatility of $\sigma_{\text{imp}} = .3$ at $r = .05$). Then the portfolio

$$\pi(S,t,c) = V(S,t) - cW(S,t) \tag{5.8}$$

satisfies

$$\mathcal{L}(\sigma, r)\pi = 0$$

$$\pi(0,t,c) = 0$$

$$\pi(S,0,c) = (S-100)^+ - c(S-90)^+.$$

The boundary condition on the barrier $X = 120$ is not known because $W(S,t)$ is not known. However, for all c we have the approximations

$$\min\{-cW_0(X,t), -cW_1(X,t)\} \le \pi(X,t,c) \le \max\{-cW_0(X,t), W_1(X,t)\}$$

where W_0 and W_1 are the BSB bounds on W. We can now define problems for functions $\widehat{\pi}_0$ and $\widehat{\pi}_1$ which bound π below and above on $S = X$, and use the BSB equations to find a greatest lower and a least upper bound on $V(S(T),T)$ in terms of $\widehat{\pi}_0(S(T),T,c)$ and $\widehat{\pi}_1(S(T),T,c)$. Computed values for $E_\ell(c)$ and $E_u(c)$ are shown in Fig. 5.4 for c increasing from -1 to $+1$ in steps of $\Delta c = .05$. We obtain the numerical values

$$V_0(S(T),T) = 0.152 \text{ at } c = 0 \quad \text{(i.e. no static hedge)}$$

and

$$V_1(S(T),T) = 3.53 \text{ around } c = -.20 \quad (W \text{ held long in the portfolio}).$$

For the unhedged option we obtain

$$V_1(S(T),T) = 3.563$$

so that static hedging does not help to narrow the envelope in this case.

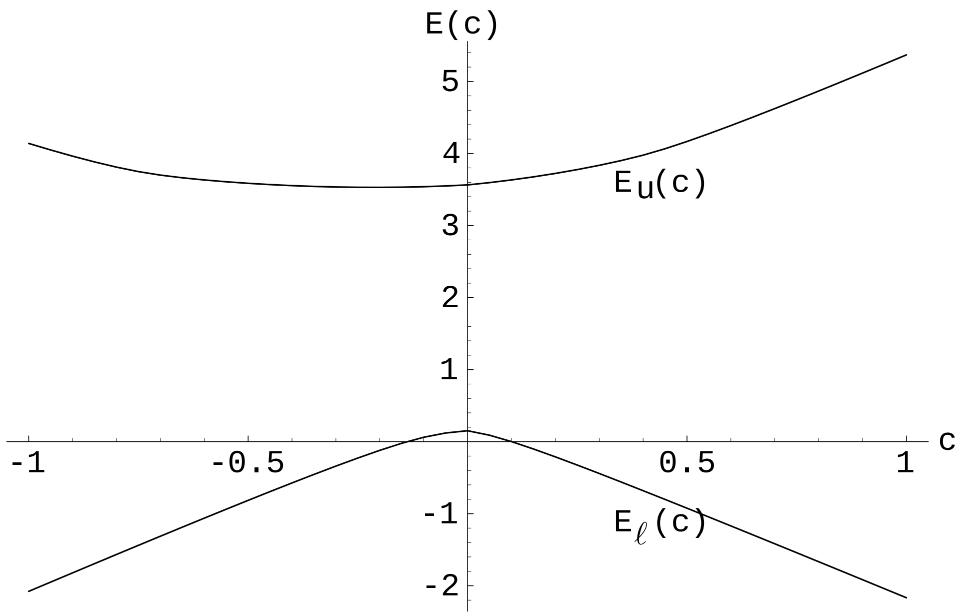

Fig. 5.4. $E_u(c)$ and $E_\ell(c)$ for approximation of the portfolio (5.8)

Appendix

All numerical solutions of the Black Scholes Barenblatt equations given above were obtained from the equations (2.7), (2.8) and their extensions (3.8), (3.9). For a fixed interest rate r the equation (2.7) is identical to the Leland model for options with transaction costs considered in [8] and the algorithm employed there was applied here. It consists of the following steps.

The equation

$$\mathcal{L}(\bar{\sigma}, \bar{r})V = F(S, t, V, V_S, |V_{SS}|) \tag{A.1}$$

with given initial condition $V(S, 0)$ and fixed or free boundary conditions is solved as a sequence of time discrete problems

$$\mathcal{L}_n(\bar{\sigma}, \bar{r})V^n = F(S, t_n, V^n, V_S^n, |V_{SS}^n|) + g_n(S) \tag{A.2}$$

subject to the boundary conditions imposed on V at time t_n. The source term $g_n(S)$ arises from the replacement of V_t in (A.1) with a backward difference formula. The time implicit formulation does not require the smoothing of initial/boundary data necessary for the application of the maximum principle and no smoothing was used in any of our numerical examples.

For all the numerical examples discussed above we used the simple first order backward difference formula

$$V_t(S, t_n) \cong g_n(S) = \frac{V^n(S) - V^{n-1}(S)}{\Delta T} \tag{A.3}$$

with constant Δt. However, for efficiency a second order time discretization should be used. The second order three level backward difference formula

$$V_t(S, t_n) \cong g_n(S) = \frac{3}{2}\frac{V^n(S) - V^{n-1}(S)}{\Delta t} - \frac{1}{2}\frac{V^{n-1}(S) - V^{n-2}(S)}{\Delta t} \tag{A.4}$$

is as easy to implement and as fast to execute as the first order formula but provides a dramatic improvement in the time resolution as shown below.

We shall denote the nonlinear ordinary differential equation (A.2) symbolically by

$$\mathcal{L}_n(\bar{\sigma}, \bar{r})V = H(S, V). \tag{A.5}$$

At each time level this equation is solved iteratively with a simple fixed point iteration

$$\mathcal{L}_n(\bar{\sigma}, \bar{r})V_{k+1} = H(S, V_k), \qquad k = 0, 1, \tag{A.6}$$

subject to the fixed and free boundary conditions which describe the option. For the initial guess $V_0(S)$ we usually choose the zero function or the converged solution from the preceding time level. A convergence proof for this iteration is given in [8] for a model problem representative for the Black Scholes equation with transaction costs. The number of iterations required for convergence decreases from four or five early on to two iterations as diffusion smoothes the initial data.

Problem (A.6) for each k corresponds to a time discretization of a standard linear Black Scholes equation with a known source term and can be solved with any of the many numerical methods for European and American options. The results reported here are obtained with a front tracking method based on the Riccati transformation introduced into the financial literature in [9]. This method typically resolves the spacial problem (A.5) to a high degree of accuracy so that numerical errors are primarily due to the time discretization.

To verify that our code reproduces published results and to show the importance of second order time differencing we list in Table A.1 the lower envelope value $V_0(S(T), T)$ for a European butterfly spread with pay-off

$$V(S, 0) = (S - 90)^+ - 2(S - 100)^+ + (S - 110)^+. \tag{A.7}$$

Extensive numerical experiments with a finite difference approximation of the BSB equation (2.5) for this option (and others) as well as a detailed convergence analysis are reported in [10] and serve to compare results.

Table A.1. Values of $V_0(S(T), T)$ for the butterfly spread (A.7)
$T = .25$, $S(T) = 100$, $r = .1$, $.15 \leq \sigma \leq .25$

Δt	with (A.3)	with (A.4)[1]	[10] first order	[10] second order[2]
.25/25	2.3533	2.2971	2.3501	2.2986
.25/50	2.3255	2.2972	2.3250	2.2981
.25/100	2.3111	2.2969	2.3116	2.2982
.25/200	2.3042	2.2972	2.3047	2.2978
.25/400	2.3009	2.2973	2.3012	2.2977

[1] Difference formula (A.3) is used at the first time level, followed by (A.4). Riccati method for (A.6) on $[2 < S < 302]$.

[2] Difference formula (A.3) is used for the first two time levels, followed by a Crank-Nicolson finite difference approximation of (2.5).

Table A.1 points out once again that any production code for option pricing should be second order accurate in time.

Compared to the Black Scholes Barenblatt equations (2.5), (2.6) the equivalent formulation (2.7), (2.8) is quite traditional and should not need the concept of a viscosity solution, particularly in view of the proof in [8] that the time discrete equation (A.5) has a twice continuously differentiable solution.

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