NUMERICAL INVESTIGATION OF EARLY EXERCISE IN AMERICAN PUTS WITH DISCRETE DIVIDENDS

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Abstract

It is well known that early exercise of an American put may not be optimal for some time before the asset goes ex dividend. This in turn implies that the early exercise boundary is not as smooth as for a put with a constant dividend yield. It is the purpose of this paper to illustrate that a straightforward numerical implementation of the time discrete method of lines for the Black Scholes equation can readily cope with the disappearance and reappearance of the early exercise boundary. The performance of the method is illustrated by computing option prices when dividends are paid discretely at a known rate or known amount, as well as with a constant dividend yield.

Key words: Black Scholes equation, American put, discrete dividend, early exercise boundary, numerical computation, method of lines.

1. Introduction

When numerical methods for the Black Scholes equation are proposed the American put with or without a constant dividend yield is a common test problem for demonstrating the performance of the method. While a problem of practical importance, it also is a problem which sidesteps the numerical difficulties inherent in the Black Scholes formulation. In particular, the early exercise boundary for an American put with a constant dividend yield is continuous and does not fall below its time asymptotic value. Hence the Black Scholes equation has to be solved only on a well defined and smoothly changing domain. Time explicit and implicit numerical methods cope well with this setting.

However, many assets pay dividends at discrete times which markedly influence the value of an option and its early exercise [4]. It is well known how to incorporate discrete dividends, either at a known rate or of a known size, into the Black Scholes formulation. It also is well known that discrete dividends completely alter the shape of the early exercise boundary [6]. The pricing of puts for assets with known discrete dividends appears well in hand ([1]; see also the comments below), but there appears to be relatively little discussion of how to actually find the boundary in the presence of discrete dividend payments. This may be due, in part, to the fact that early exercise may no longer be optimal over certain time intervals or may occur at very low asset values. Now the convective term of the diffusion equation becomes important which complicates its numerical solution (see the discussion in [11]). In addition, a discontinuous early exercise boundary has to be found which in general requires a time implicit numerical method.

It is the purpose of this paper to describe how American options with discrete dividend payments can be priced, after minor technical modifications, with the method of lines which has been applied earlier to Black Scholes and related models for American options without discrete dividends [2], [7]. This method, when coupled with a numerical integration of the resulting ordinary differential equations, applies to a general diffusion equation. It is fully time implicit, non-iterative and basically unaffected by the presence of an early exercise boundary which is computed explicitly. We shall use this method to price an American put with a proportional or fixed dividend payment. Our particular goal is to demonstrate that the quantitative arguments and qualitative pictures of [6] for the free boundary are, in general, consistent with a highly accurate numerical solution of the Black Scholes model.

2. The model equations and solution method

We shall consider a standard American put in the Black Scholes context with strike price K and maturity T and, for simplicity, only one dividend payment at time t_d. It is well known that on $[t_d, T)$ and $[0, t_d)$ the value $P(S,t)$ of the option is the solution of the Black Scholes equation

$$\frac{1}{2}\sigma^2 S^2 P_{ss} + (r - \rho_c) S P_s - rP + P_t = 0$$

where S is the value of the underlying, t and T are time and maturity, where σ is the volatility and where r denotes the risk-free interest rate. ρ_c is the dividend yield for a continuously paid dividend, should such a payment be present.

The put satisfies the boundary conditions

$$\lim_{S \to \infty} P(S, t) = 0$$

and the early exercise condition

$$P(S(t), t) = K - S(t)$$

$$P_S(S(t), t) = -1$$

where $S(t)$ is the unknown free boundary, provided it exists.

At maturity we have the final condition

$$P(S, T) = \max\{K - S, 0\}$$

while the dividend payment is accounted for by the interface condition

$$P(S, t_d-) = P(f(S), t_d+)$$

where, for example,

$$f(S) = (1 - \rho)S$$

if the dividend is paid at a rate ρ, or where

$$f(S) = S - D$$

if a payment of magnitude D is made.

We remark that this particular application is often resolved with an approximation due to Barone-Adesi and Whaley which relies on balancing the interest income on the early exercise return against the increase in the value of the put at the ex-dividend date [1]. It is not clear how these considerations can be derived from the Black-Scholes equation but the predicted option prices are remarkably consistent with the numerical solution of the Black Scholes equation for the American put presented in [1] and below. The Barone-Adesi Whaley method is attractive because it is observed to be 2000 times faster than the numerical method against which it is checked but which is not further identified [1]. However, hardware and software have changed in the intervening ten years. It may well be that a direct numerical integration of the Black Scholes equation for assets with discrete dividends has become competitive because machines and algorithms are faster. The dominant advantage of a numerical method is that it provides an accurate solution of the model equation, whether Black Scholes or any variation thereof. For example, while we consider here dividends paid at a known rate or of a known amount our algorithm does not depend on the actual form of $f(S)$ so a more complicated scheme for the loss of value of the underlying at the ex-dividend date could be chosen [5].

It is convenient to scale the problem by $u(x,t) = P(S/K,t)/K$ where $x = S/K$ so that the strike price is normalized to 1. Then the following free boundary problem results

$$\frac{1}{2}\sigma^2 x^2 u_{xx} + (r-\rho_c)xu_x - ru + u_t = g(x,t).$$

$$\lim_{x\to\infty} u(x,t) = 0$$

$$u(s(t),t) = 1 - s(t)$$

$$u_x(s(t),t) = -1$$

$$u(x,T) = \max\{1-x, 0\}$$

and

$$u(x,t_d-) = u(f(Kx)/K, t_d+).$$

For the description of the algorithm we have added a source term to the scaled Black Scholes equation. Such source terms arise, e.g., when random jumps are added to the Brownian motion of the underlying [8], when the nonlinear Black Scholes equation with transaction cost of Leland type is solved iteratively [9], or when Greeks like the rho and vega of an American option must be found. The method of lines approach (in the context of free boundary problems for a general variable coefficient diffusion equation) involves approximating the time-continuous problem by a sequence of time discrete ordinary differential equations. If we choose a first order implicit approximation defined at time level

$$t_n = t_d + n\Delta t, \qquad n = 0,..,N, \qquad \Delta t = (T-t_d)/N$$

then we repeatedly solve

$$Lu_n \equiv \frac{1}{2}\sigma^2 x^2 u_n''(x) + (r-\rho_c)xu_n'(x) - \left(r + \frac{1}{\Delta t}\right)u_n(x) = -\frac{1}{\Delta t}u_{n+1}(x) + g(x,t_n) \tag{2.1}$$

$$u_n(X) = 0 \tag{2.2}$$

$$u_n(s_n) = 1 - s_n \tag{2.3}$$

$$u_n'(s_n) = -1 \tag{2.4}$$

where

$$u_N(x) = \max\{1-x, 0\}, \qquad s_N = 1. \tag{2.5}$$

We point out the the condition at $S \to \infty$ has been replaced by the up and out barrier condition (2.2). The influence of X can be checked experimentally or estimated with the Black Scholes formula [7]. Alternatively, a reflection condition can be implemented (see also [7]).

An analogous approximation is used for propagating the time discrete solution from $t = t_d$ to $t = 0$. If we restart our count and use N time steps over $[0, t_d)$ then

$$\Delta t = t_d/N, \qquad t_n = n\Delta t$$

and (2.5) is replaced by

$$u_N(x) = u_0(f(Kx)/K) \tag{2.6}$$

where u_0 is the computed answer at t_d. The goal is to find the value of the put

$$P(S,0) = Ku_0(S/K)$$

where now u_0 is the final solution obtained from the integration over $[0, t_d)$.

The time discrete approximation (2.1) is known as a method of lines approximation of the diffusion equation. For a put without dividends it has a recursive analytic solution on which efficient numerical methods can be based [2]. Here in order not to be locked into a constant coefficient Black Scholes model we shall solve the ordinary differential equation (2.1) on the unknown interval $[s_n, X]$ numerically for u ($\equiv u_n$) and s_n with a Riccati transformation method [7]. We use the fact that

$$u(x) = R(x)v(x) + w(x) \tag{2.7}$$

where $v(x) = u'(x)$ and where R and w are solutions of the well defined initial value problems

$$R' = 1 + \frac{2(r - \rho_c)}{\sigma^2 x} R - \frac{2}{\sigma^2 x^2}\left(r + \frac{1}{\Delta t}\right) R^2, \qquad R(X) = 0 \tag{2.8}$$

$$w' = -\frac{2}{\sigma^2 x^2}\left(r + \frac{1}{\Delta t}\right) R(x)w + \frac{2}{\sigma^2 x^2} R(x)\left[u_{n+1}(x) - g(x, t_n)\right], \qquad w(X) = 0. \tag{2.9}$$

These equations apply specifically to the standard Black-Scholes equation (2.1). They are readily modified to allow deterministic volatility surfaces and variable interest rates or to model interest rate derivatives. For their derivation and an application to a CEV model and an implied volatility function calculation we refer to [7].

The free boundary s_n at time t_n is a root of the equation

$$\phi(x) \equiv (1 - x) - R(x)(-1) - w(x) = 0. \tag{2.10}$$

Each such root yields a solution of (2.1). Should multiple roots be present (not usually the case) it is generally simple to choose the appropriate s_n on the basis of financial or continuity considerations. Once s_n is found the solution is completed by integrating

$$v' = \frac{2}{\sigma^2 x^2}\left[\left(r + \frac{1}{\Delta t}\right) R(x)v - (r - \rho_c)\, xv + \left(r + \frac{1}{\Delta t}\right) w(x) - \frac{1}{\Delta t}\, u_{n+1}(x) + g(x, t_n)\right]. \tag{2.11}$$

$$v(s_n) = -1$$

If at time level t_n early exercise is not optimal, as in the case of a fixed sum dividend payment, then the equation (2.10) does not have a solution. If early exercise is not permitted at time t_n then equation (2.10) is ignored altogether. In either case (2.11) is integrated from $x_0(t_n)$ to X subject to the initial condition

$$v(x_0(t_n)) = [u(x_0(t_n)) - w(x_0(t_n))]/R(x_0(t_n))$$

which is obtained from (2.7). Here $x_0(t)$ is a known lower bound on the feasible values of the underlying where $u(x_0(t), t)$ is also assumed to be known. (If $x_0(t) = 0$ then this boundary condition will have to be approximated by a barrier condition at some $x_0 > 0$ since in general the solution of the Riccati equation satisfies $R(0) = 0$.) The function v obtained by integrating (2.11) is substituted into (2.7) and yields the scaled value of the put at $t = t_n$ over $[s_n, X]$. Below s_n the put assumes its intrinsic value. We note that for constant coefficients the Riccati equation does not depend on n and needs to be solved only once.

In the absence of discrete dividend payments the scaled early exercise boundary $s(t)$ and its numerical approximation decay from $s(T) = 1$ to the steady state solution

$$s_\infty = \frac{\gamma}{\gamma + 1}$$

where γ is the positive root of

$$\frac{1}{2}\sigma^2\gamma(\gamma - 1) + (r - \rho_c)\gamma - r = 0.$$

It follows that $s_{n+1} \geq s_n \geq s_\infty$. For realistic financial parameters $s_\infty \gg 0$ so that the coefficients and the source term in (2.8, 9, 10) remain uniformly bounded on $[s_n, X]$. On the other hand, if dividends are paid at discrete times then early exercise may not be optimal so that the differential equations must be solved on $(0, X)$ where the coefficients are no longer bounded. This introduces some minor technical complications into the implementation of the algorithm as presented in [7] for a standard put. They will be discussed in the Appendix where the numerical solution of (2.7–11) is outlined.

3. Discrete dividend payments

Case 1: Known dividend rate: Let us first consider the case of a dividend payment at time t_d made at a known rate ρ. Since $u(x, t_d+)$ is decreasing and convex in x on

$[s(t_d+), \infty)$ and linear on $[0, s(t_d+)]$ it follows that

$$(3.1) \qquad u(x, t_d-) = u((1-\rho)x, t_d+)$$

satisfies

$$u(x, t_d-) > \max\{1-x, 0\}, \qquad x > 0.$$

Hence

$$s(t_d-) = 0$$

which expresses the well-known observation that early exercise is not optimal just prior to the ex-dividend date. However, for any nonzero time interval Δt before the ex-dividend date there will be an optimal boundary $s(t_d - \Delta t) > 0$. In fact, the interest income from early exercise over the time span Δt balances the gain of the value of the put at the ex-dividend date when the (scaled) asset price satisfies

$$\exp(r\Delta t) - 1 = \rho s(t_d - \Delta t) \equiv \rho \left(s(t_d - \Delta t) - s(t_d-) \right)$$

which suggests that the slope of the early exercise boundary is given by

$$(3.2) \qquad \lim_{t \to t_d-} s'(t) = -r/\rho.$$

This slope is independent of the volatility σ. The numerical results of Fig. 1b suggest that (3.2) is correct.

Case 2: Known dividend payment: When the asset pays a known dividend D then the initial condition analogous to (3.1) becomes

$$(3.3) \qquad u(x, t_d-) = u(x - D/K, t_d+).$$

Note that in this case the strike price K does not scale out of the model equations. It follows again that $u(x, t_d-) > \max\{1-x, 0\}$. The relation (3.3) would require pricing a put for negative values of the underlying when $x < D/K$. However, for a known dividend D the asset value should not fall below the time discounted value of D. Hence as long as early exercise is not optimal the Black-Scholes equation must be solved on the time varying but known interval $[x_0(t), X]$ where

$$x_0(t) = D/Ke^{-r(t_d-t)}.$$

The value of the put is given by

$$u(x_0(t), t) = 1 - D/Ke^{-r(t_d - t)}.$$

It is argued [1], [6] that as long as the dividend payment D exceeds the risk free interest obtained from early exercise before the ex-dividend date an early exercise is not optimal. Both returns balance when

$$\exp(r(t_d - t)) = 1 + D/K$$

which suggests that the early exercise boundary should disappear at time

$$t^* = t_d - \ln(1 + D/K)/r. \tag{3.4}$$

This relation is independent of the model for the option price, but our numerical results from the Black Scholes equation are remarkably consistent with this prediction which again is independent of σ. Unfortunately, at present there are no analytical results on the location $s(t^*)$ of the early exercise boundary and its smoothness at the time of disappearance. However, our numerical results consistently show a jump at t^* and do not reproduce the continuous decay of the graphs in [6, p. 155, 157].

4. Numerical results

For illustration we shall consider an American put with maturity $T = .5$ years and an ex-dividend date of $t_d = .3$ years. As standard data we choose

$$r = .08, \qquad \rho_c = 0 \quad \text{and} \quad \sigma = .4$$

In all calculations the up and out barrier is placed at three times the strike price, i.e., $X = 3$. Details of the numerical solution of the method of lines equations are given in the Appendix. We only want to point out here that the computational results shown below come from a fine space and time discretization of the model and that the results remain stable as the meshes are coarsened or further refined. Further comments are found in the Appendix.

Case 1: Dividend rate $\rho = .02$. Fig. 1a shows the early exercise boundary as a function of $\tau = T - t$ when the discrete dividend is modeled by (3.1). The early exercise boundary decays linearly with precisely the rate predicted by (3.2) as $t \to t_d$. Fig. 1b shows the early exercise boundary for volatilities of $\sigma = .01$, $\sigma = .4$ and $\sigma = .8$. The slope of the exercise boundary near t_d apparently is not a function of σ. Fig. 1c shows the scaled put, delta and gamma at $t = 0$ (recall that $P(S,t) = Ku(S/K,t)$, $P_S(S,t) = u_x(S/K,t)$, $P_{SS}(S,t) = u_{xx}(S/K,t)/K$). We note that theoretically

$$u_{xx}(s(0),0) = 2r/(\sigma s(0))^2.$$

For our computed value of $s(0) = .6584$ this formula predicts $u_{xx}(s(0),0) = 2.3068$. The numerical value in the graph is gamma $= 2.307$.

For comparison we show in Fig. 1a also the early exercise boundary found without a discrete dividend payment but with a constant dividend yield of

$$\rho_c = .0484$$

which was determined with a bisection method such that the value of the put at $t = 0$ for $x = 1$ duplicates the result for the discrete dividend case. Table 1 shows some numerical values of the put for discrete and constant dividend yield payments.

Table 1a. Values of the normalized American put at $t = 0$

x	$\rho = .02$	$\rho_c = .0484$
.8	.2194	.2200
1.0	.1034	.1034
1.2	.0429	.0432

It is seen from Fig. 1a that the early exercise boundary at $t = 0$ is roughly the same for both cases while at a time like $t = .28$ they are far apart. Corresponding numerical values for the put at $t = .28$ are given below.

Table 1b. Values of the normalized American put at $t = .28$

x	$\rho = .02$	$\rho_c = .1093$
.8	.2168	.2106
1.0	.0764	.0764
1.2	.0184	.0188

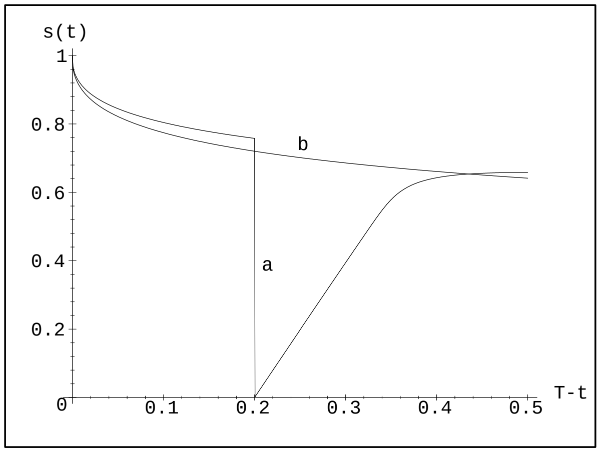

Fig. 1a: Early exercise boundary as a function of $\tau = T - t$;
(a) one discrete dividend payment ρS at $t_d = .3$; (b) a constant dividend yield.

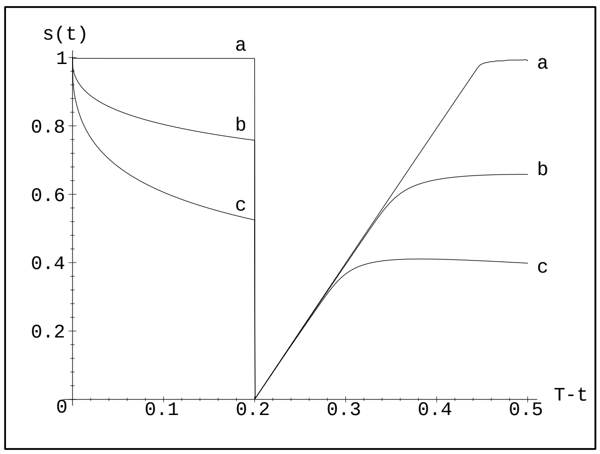

Fig. 1b: Influence of the volatility on the early exercise boundary of a put with one dividend payment ρS; (a) $\sigma = .01$; (b) $\sigma = .4$; (c) $\sigma = .8$.

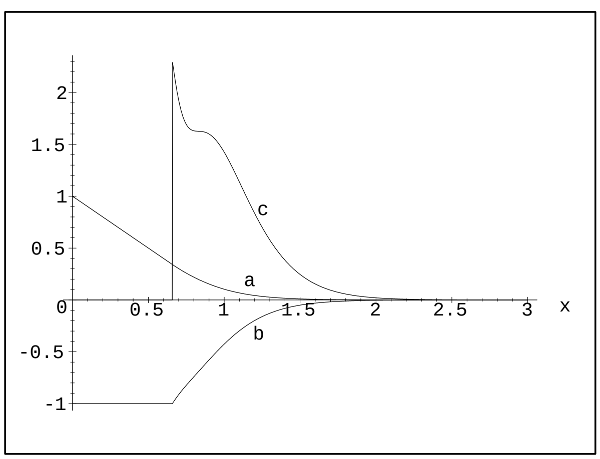

Fig. 1c: Plot of the scaled price (a), delta (b) and gamma (c) at $t = 0$ for an American put with one dividend payment ρS.

The prices of the put are again roughly consistent, but the early exercise boundary from the constant dividend yield is meaningless if discrete dividends are paid.

Case 2: Dividend payment $D/K = .02$. We shall carry out numerical experiments analogous to the above when a known dividend is paid on the asset. We note that this choice of dividend is the same as the dividend paid at the known rate at $x = S/K = 1$.

Fig. 2a shows the early exercise boundary as a function of $\tau = T - t$. Early exercise is not optimal over the period

$$\sim .052 < t \leq .3.$$

Around $t = .052$ the early exercise boundary jumps spontaneously to $x_0(t) = D/Ke^{-r(t_d - t)} = .0196$. This behavior is quite independent of the mesh parameters as the following data from successively finer meshes show.

$s(t)$	$\Delta t = 5 \cdot 10^{-4}$	$\Delta t = 2.5 \cdot 10^{-4}$	$\Delta t = 1.25 \cdot 10^{-4}$
$s(.052500)$	.0196	.0196	.0196
$s(.052375)$			.4324
$s(.052250)$		.4523	.4531
$s(.052125)$			.4650
$s(.052000)$	.4714	.4727	.4734

We do not know whether $s(t)$ is continuous at this time or how to predict the location to which the boundary jumps but note that the time of its jump is quite consistent with the estimate (3.4) which for our data is

$$t^* = t_d - \ln(1 + .02)/.08 = .05247.$$

In fact, the computed time of disappearance of the free boundary appears to converge to this theoretical value as $\Delta t \to 0$. The difference in the time of the jump between the coarse and the fine time step calculations amounts to 3.3 hours for the option with $T = .5$ years.

The predicted time of the jump of $s(t)$ in (3.4) is independent of the volatility. The numerical solution likewise appears to be independent of σ at this time as may be inferred from Fig. 2b which shows the early exercise boundaries for $\sigma = .01$, $\sigma = .4$ and $\sigma = .8$.

A plot of the option price u, the delta u_x and the scaled gamma u_{xx} at $t = 0$ is shown in Fig. 2c. The theoretical value of gamma for the computed early exercise boundary $s(0) = .6247$ is $u_{xx}(s(0), 0) = (2r/(\sigma s(0))^2 = 2.5627$. The numerical value in Fig. 2c is gamma $= 2.549$. The local minimum in the gamma near the early exercise boundary is absent when there are no discrete dividends.

As before we also compute the put without a discrete dividend but with a constant dividend yield $\rho_c = .0552$ chosen by trial and error such that the put at $x = 1$ has the same value at $t = 0$ as the above put with the discrete dividend $D/K = .02$. The corresponding early exercise boundary is contrasted to that for the discrete dividend in Fig. 2a. Some numerical values for the put are given in Table 2a.

Table 2a. Values of the normalized American put at $t = 0$

x	$D/K = .02$	$\rho_c = .0552$	
.8	.2228	.2207	(.2280)
1.0	.1046	.1046	(.1040)
1.2	.0430	.0439	(.0421)

If the option values are to be matched at $x = 1$ and $t = .28$ a constant yield of $\rho_c = .1106$ must be used. In this case the following values for the option are obtained.

Table 2b. Values of the normalized American put at $t = .28$

x	$D/K = .02$	$\rho_c = .1106$	
.8	.2205	.2108	(.2208)
1.0	.0765	.0765	(.0764)
1.2	.0179	.0189	(.0178)

Finally we point out that if in analogy to European options our options are valued as American puts without dividend but at a value of the underlying discounted by the present value of the dividend (see, e.g., [3]) then for both $t = 0$ and $t = .28$ the prices shown in parentheses in Table 2a,b are obtained. The option prices at the money agree quite well with those from the discrete dividend calculation. However, the early exercise boundary of any approximation without a discontinuity of the underlying asset value at t_d is meaningless and over long time spans will misprice the option.

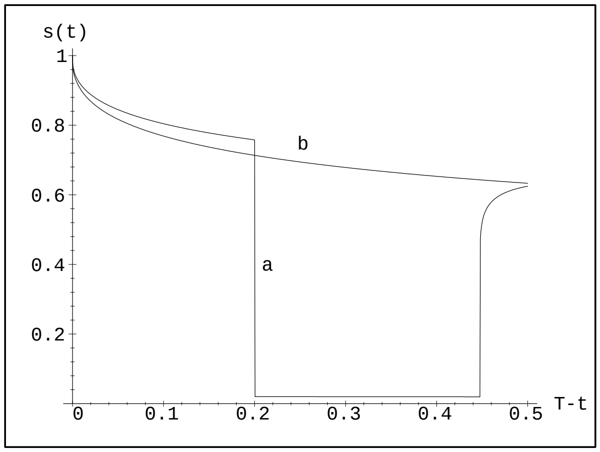

Fig. 2a: Early exercise boundary as a function of $\tau = T - t$;
(a) one discrete dividend payment D at $t_d = .3$; (b) a constant dividend yield.

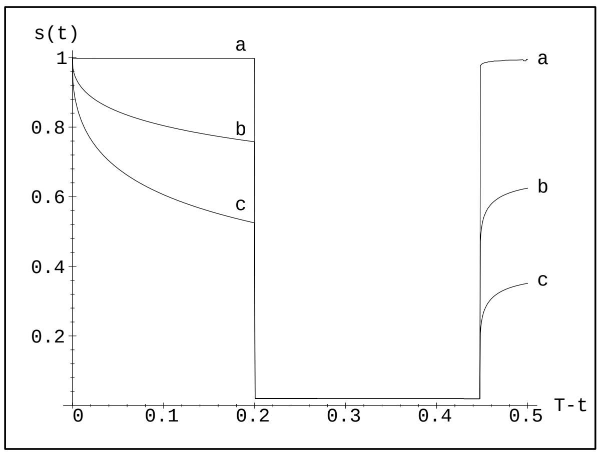

Fig. 2b: Influence of the volatility on the early exercise boundary of a put with one dividend payment D; (a) $\sigma = .01$; (b) $\sigma = .4$; (c) $\sigma = .8$.

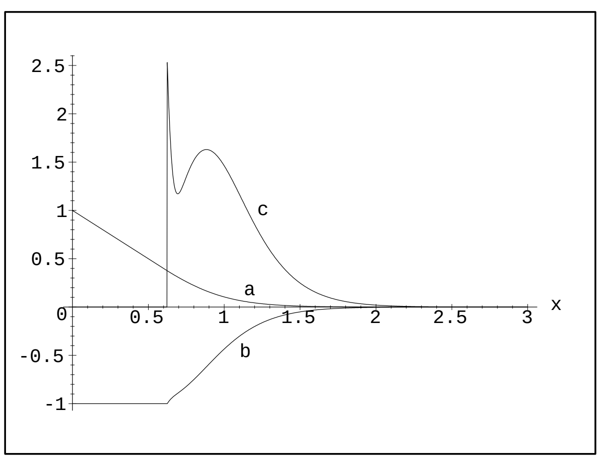

Fig. 2c: Plot of the scaled price (a), delta (b) and gamma (c) at $t = 0$ for an American put with one dividend payment D.

Summary and Outlook

A non-iterative, numerical front tracking method is applied to a time discrete approximation of the Black Scholes diffusion equation to compute the early exercise boundary for an American put when the asset pays dividends at discrete times. Financial considerations predict a volatility independent discontinuous behavior of the early exercise boundary near the ex-dividend date. The numerical results quantitatively mirror this behavior. However, analytic results for the free boundary near ex-dividend times remain lacking.

The numerical method in this study applies to general variable coefficient diffusion equations and can handle equity and interest rate derivatives. Moreover, in conjunction with a line iteration it has been used to solve certain multi-dimensional parabolic free surface problems of mathematical physics. Its applicability to Asian and basket options is currently under study [10].

Appendix - Numerical Considerations

At every time level the solutions $R(x)$, $w(x)$ and $v(x)$ of the initial value problems (2.8, 9, 11) are needed. They can be computed numerically to an arbitrary degree of accuracy provided that they are smooth and bounded. In view of the unbounded coefficients on $(0, X)$ this property is not obvious. However, one can show that these functions are indeed smooth and bounded on $[0, X]$.

Let us consider the Riccati equation (2.8). Its solution is non-positive on $[0, X]$. Since the right hand side is infinitely differentiable in R and x the solution exists and is smooth on any interval $(a, X]$ on which it exists. It is a simple calculation to show that there exists a positive constant $K(r, \rho_c, \sigma)$ such that the function

$$f(x) \equiv K\sqrt{\Delta t}x + R(x)$$

satisfies $f'(x^*) < 0$ whenever $f(x^*) = 0$. Hence $f(x) > 0$ on $(0, X]$ so that $0 > R(x) > -K\sqrt{\Delta t}x$ which insures that R is smooth and uniformly bounded on $(0, X]$. We also note that $R(x)$ has to decay exactly linearly since $R(x) = O(x^c)$, $c > 1$ implies that $\lim R'(x) = 1$ which is inconsistent with $R(x) < 0$ on $(0, X)$. The smoothness and boundedness of $R(x)$ insure that the Riccati equation can be solved to arbitrary accuracy by any convergent numerical integrator.

The linear equations (2.9,11) have exponentially decreasing fundamental solutions in the direction of integration and are simple to solve over any interval $[a, X]$ with $a > 0$. For the case of a discrete dividend payment D we know a priori that

$$a \geq x_0(t) = D/Ke^{-r(t_d - t)}$$

so that w and v likewise can be found to any degree of accuracy by choosing a fine spacial mesh. For the case of a discrete payment at rate ρS we note that at time level $t_{N-1} = t_d - \Delta t$ the source term in (2.1) on $[0, s(t_d)/(1-\rho)]$ is given as

$$u_N(x) = 1 - (1-\rho)x.$$

As is well known, and exploited in the analytic method of lines [2], this implies that on this interval $u_{N-1}(x)$ has the structure

$$u_{N-1}(x) = c_1 x^{\gamma_1} + c_2 x^{\gamma_2} + 1/(1 + r\Delta t) - (1-\rho)x \tag{A.1}$$

where γ_1 and γ_2 are the positive and negative roots of

$$\frac{1}{2}\sigma^2\gamma(\gamma - 1) + (r - \rho_c)\gamma - (r + 1/\Delta t) = 0.$$

c_1 and c_2 are coefficients to be determined such that u_{N-1} satisfies the boundary conditions at $x = x_0(t_d - \Delta t)$ and such that it links up smoothly with the representation of $u_{N-1}(x)$ over $[s(t_d)/(1-\rho), X]$. We note that $|\gamma_i| = O(1/\sqrt{\Delta t})$ so that $|\gamma_i| \gg 1$. We now use the observation that w is independent of the boundary condition imposed at $x_0(t_d - \Delta t)$. If we set

$$u_{N-1}(0) = 1/(r\Delta t + 1)$$

then it follows from (A.1) that $c_1 = 0$ and that $\lim_{x \to 0} v(x) = -(1-\rho)$. This in turn implies that

$$\lim_{x \to 0} w(x) = 1/(1 + r\Delta t), \qquad \lim_{x \to 0} |w'(x)| < \infty$$

which guarantees that the function $\phi(x)$ of (2.10) satisfies

$$\lim_{x \to 0} \phi(x) > 0.$$

Since $\phi(X) = 1 - X < 0$ we have established that for any $\Delta t > 0$ there is an early exercise boundary at $t_d - \Delta t$ so that again w and v are found on an interval $[a, X]$ where $a > 0$. Again we can argue that for a given Δt the functions w and v can be found to any degree of accuracy.

The numerical integration of the differential equations is carried out on a time invariant but not necessarily uniform spacial mesh $\{x_j\}$ to which is adjoined at time t_n the free boundary s_n. As integrator for the Riccati equation (2.8) we choose the trapezoidal rule which requires only nodal values and no interpolation which would become necessary for an adaptive method. The algebraic equations arising from the trapezoidal rule for the Riccati equation lead to a quadratic equation of the form

$$a_j R_j^2 + b_j R_j + c_j(R_{j+1}) = 0$$

which can be solved for R_j with the quadratic formula.

The linear equations (2.9, 11) are also integrated with the trapezoidal rule as long as it properly accounts for the monotone decrease of the fundamental solutions. For example, if $\Delta x_j = x_j - x_{j-1}$ then oscillations will arise in the numerical values of w unless

$$1 + \frac{R_{j+1}(r\Delta t + 1)\Delta x_{j+1}}{\sigma^2 x_{j+1}^2 \Delta t} > 0. \tag{A.2}$$

A similar constraint applies to the integration of v. This requirement is most pronounced at $x_1 = x_0(t_n) + \Delta x$. But rather than refining our mesh to satisify this inequality a priori we simply switch to an implicit Euler method whenever (A.2) does not hold. In general, only a few mesh points near $x = 0$ are affected, and usually they lie to the left of s_n where R, w and v are not needed.

As the equations for R and w become available during the integration from $x = X$ to $x = 0$ the function ϕ is evaluated. If it changes sign between x_j and x_{j+1} then s_n is found as the root of the cubic interpolant of ϕ at the points $\{x_{j-1}, x_j, x_{j+1}, x_{j+2}\}$. For the integration of v from s_n to x_{j+1} the trapezoidal rule with linearly interpolated values for R, w and u_{n+1} at s_n is used.

The numerical results shown above were obtained with the following mesh on $[0, X]$

where $X = 3$:

$$\Delta x = 10^{-5} \qquad \text{on } [.0001, .01] \qquad (= 990 \text{ mesh points})$$

$$\Delta x = 2.5 \cdot 10^{-3} \qquad \text{on } [.01, 1] \qquad (= 396 \text{ mesh points})$$

$$\Delta x = 4 \cdot 10^{-3} \qquad \text{on } [1, 3] \qquad (= 500 \text{ mesh points})$$

Although [2.8, 9, 10] are computed on [.0001, 3] it should be noted that usually $s(t) > .01$ and $x_0(t) > .01$ so that the fine mesh on $[.0001, .01]$ did not actually affect our solution.

The time step was $\Delta t = 1/2000$ (i.e., 400 time steps from T to t_d and 600 time steps from t_d to 0). The above mesh sizes were chosen to insure that the numerical results are stable with respect to changes of the space and time mesh. To illustrate this point we give in Table A1 selected numerical values for the position of the free boundary plotted for the above mesh parameters in Figs. 1a and 2a, as well as the corresponding values when all space and time intervals are either doubled or halved.

Table A1. Influence of the mesh parameters on the free boundary $s(t)$

$T-t$	doubling the step size	plotted results of Fig. 1a	halving the step size
.001	.959714	.962025	.963863
.01	.913178	.913708	.914034
.1	.804301	.804313	.804301
.2	.757835	.757855	.757872
.3	.393862	.394483	.394838
.4	.642384	.642799	.642886
.5	.658309	.658421	.658503
$T-t$	doubling the step size	plotted results of Fig. 2a	halving the step size
.44	.019620	.019620	.019620
.45	.521015	.522463	.523377
.46	.578331	.579075	.579461
.47	.598726	.599295	.599543
.48	.610786	.611077	.611267
.49	.618609	.618965	.619115
.5	.624418	.624666	.624762

Only at $t = .001$ does there appear to be a notable discrepancy, which reflects the difficulty of computing the rapidly moving early exercise boundary just prior to expiry.

Plots were generated with MAPLE and represent linearly interpolated nodal values. Note that gamma is given explicitly by the right hand side of (2.11). None of the graphs were smoothed a posteriori.

A final comment on the execution times required for this algorithm. Our results were obtained in double precision with a Fortran program on a Sun Ultra 10 workstation with one processor. The code was optimized by the complier. When the program was run with half the mesh points and twice the time step the results were obtained instantenously and the time a.out command listed no execution time. For the data shown above the time a.out command returned 3.0u 0.0s. When the program was run with twice the number of mesh points and half the time step (i.e., 2000 time steps) the time a.out command returned 11.0u 0.0s. Several options exist to speed up the calculation. To begin with, numerical results are only needed to market accuracy which we believe can be obtained from a much coarser mesh and fewer time steps than used here. Secondly, earlier experiments have shown that a second order backward time difference

$$u_t(x, t_n) \cong \frac{3}{2}\left[\frac{u_{n+1} - u_n}{\Delta t}\right] - \frac{1}{2}\left[\frac{u_{n+2} - u_{n+1}}{\Delta t}\right]$$

is as easy to implement as the first order quotient used above but allows much larger time steps when $s(t)$ is smooth [7]. However, jumps in $s(t)$ would require a restart with a backward quotient for one time level. And finally, many components of the algebraic formulas coming from the trapezoidal rule remain unchanged from one time level to the next and can be precomputed and stored to reduce arithmetic operations. However, execution time was not of concern so these options were not priced.

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