A new concordance invariant of knots in sums of $S^2 \times S^1$

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Two knots $K$ and $J$ inside $S^3$ are concordant if there is a smooth, properly embedded annulus in $S^3 \times [0, 1]$ whose boundary is $K \times \{0\} \sqcup J \times \{1\}$.
Preliminaries

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**Proposition**

A knot (or link) $K \subset S^3$ is slice if and only if it is concordant to the unknot (or unlink).
When is a link trivial modulo concordance?

For oriented links $L \subset S^3$, linking number is one of the first tools we use to detect nontrivial links.
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Example

![Link Diagram]
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(a) $lk(K, J) = 1$

(b) $lk(K, J) = 0$ so ...?
Fact:

If $L$ is an $n$-component oriented link with $L_i$ the 0-framed longitude of the $i^{th}$ component of $L$ and $G = \pi_1(S^3 \setminus \nu(L), \ast)$, then

$$[L_i] = \Sigma_{i=1}^{n} \text{lk}(L_i, L_j) \cdot x_i \in H_1(S^3 \setminus \nu(L)) = G/[G,G]$$

where $x_i$ represents the $i^{th}$ meridian.
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where \( x_i \) represents the \( i^{th} \) meridian.

Question:

What if you look at the image of this longitude in a different quotient of \( G \)?
Linking number in the context of groups

**Recall:**

The lower central series of a group $G$ is defined recursively by $G_1 = G$, $G_{n+1} = [G_n, G]$. 

**Theorem (Casson '75)**

If $L_1$ and $L_2$ are concordant links whose groups are $G$ and $H$, then $G/G^q$ and $H/H^q$ are isomorphic for all $q$. 

**Motivating Idea:** Look at the image of a longitude $L_i$ inside the quotient $G/G^q$!
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Concordance data from the lower central series

Notice:
If $L \subset S^3$ is an $n$-component link, then $H_1(S^3 \setminus \nu(L)) = G/[G, G] = \mathbb{Z}^n$ and the $n$-component unlink has $\pi_1(S^3 \setminus \nu(U), \ast) \cong F(n)$.
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To compute higher order linking numbers (called Milnor’s invariants) back in ‘54:

Find clever presentation of \( G/G_q \).

Write \( i \)th longitude modulo \( G_q \) as a word in meridians (one for each component).

Use the Magnus embedding to map this word to a power series ring in \( n \) non-commuting variables and read off coefficients of degree \( q-1 \) terms modulo coefficients of lower order terms.
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Rough definition (Milnor ‘54)

The Milnor invariants of an $n$-component link $L \subset S^3$ with link group $G$ are a set of integers

$$\bar{\mu}_L(I) \in \mathbb{Z}$$

with $I = (i_1...i_k)$ and $i_j \in \{1, ..., n\}$ detecting when $G/G_q$ stops being isomorphic to $F/F_q$ where $F$ is the rank $n$ free group.
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- $\bar{\mu}_L(ij) = lk(L_j, L_i)$

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- $\bar{\mu}_L(ij) = lk(L_j, L_i)$
- $\bar{\mu}_L(ijk) = \text{triple linking number}$

$$\bar{\mu}_L(ijk) = 1$$
Why are $\bar{\mu}$-invariants useful?

- (Milnor ‘54) $\bar{\mu}_L(I)$ is a link homotopy invariant for each $I$ with non-repeating indices.
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- (Turaev ‘79, Porter ’80) $\bar{\mu}_L(I)$ can be computed by evaluating Massey products in $H^1(S^3 \setminus \nu(L))$ on individual boundary components.
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- (Turaev ’79, Porter ’80) $\bar{\mu}_L(I)$ can be computed by evaluating Massey products in $H^1(S^3 \setminus \nu(L))$ on individual boundary components.
- (Cochran ’90) The first non-zero $\bar{\mu}_L(I)$ (and thus, the first $q$ for which $G/G_q$ is not isomorphic to $F/F_q$) can be computed using intersection theory.
Computing $\bar{\mu}_L(I)$

**Example**

We can detect non-zero Milnor's invariants by intersecting surfaces and computing linking numbers of the intersection curves.
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(a) $\{C'(x), C'(y)\} \in \mathcal{C}$
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Build a surface system $(C, V)$

(a) $\{C(x), C(y)\} \in C$

(b) $V(x) \in V$

(c) $V(y) \in V$
Computing $\bar{\mu}_L(I)$

Throw intersection curves in $\mathcal{C}$

$V(x) \cap V(y)$ in a simple closed curve $c(xy)$.
Computing $\tilde{\mu}_L(I)$

Throw intersection curves in $C$

$V(x) \cap V(y)$ in a simple closed curve $c(xy)$.

Compute pairwise linking numbers of curves in $C$

$lk(C(xy), C^+(xy)) = -1$
Computing $\bar{\mu}_L(I)$

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$\text{lk}(C(xy), C^+(xy)) = -1$ which indicates (by work of Cochran) that $L$ has a nonzero $\bar{\mu}_L(I)$ of weight $|I| = 4$ (and thus $G/G_5$ is not isomorphic to $F/F_5$).
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$lk(C(xy), C^+(xy)) = -1$ which indicates (by work of Cochran) that $L$ has a nonzero $\bar{\mu}_L(I)$ of weight $|I| = 4$ (and thus $G/G_5$ is not isomorphic to $F/F_5$). If all possible linkings are trivial, run the process again.
Is there a version of this linking data for knots or links in other 3-manifolds?

**Question:**

For a knot or link $L \subset M$ where $M$ is an oriented 3-manifold, can we similarly extract concordance data from quotients of $G = \pi_1(M \setminus \nu(K), \ast)$ by $G_q$?

**Previous results:**

(D. Miller '95) Defined Milnor’s invariants for knots homotopic to a singular fiber in a Seifert fiber space using covering spaces and combinatorial group theory.

(Heck '11) Defined a homotopy-theoretic version of Milnor’s invariants for knots in prime manifolds.

**Idea:**

Exploit surfaces to define analogue of first non-vanishing $\bar{\mu}_1(L)$. 
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Idea:
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A half grope of class 3.
Realizing iterated commutators geometrically

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A class $n$ half-grope is a 2-complex made of $n - 1$ layers of surfaces.

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2. Exactly half of the generators in a symplectic basis for $H_1(\Sigma_2)$ bound surfaces $\Sigma_3^i$ where $1 \leq i \leq g(\Sigma_2)$.
3. For each $i$, exactly half of the generators in a symplectic basis for $H_1(\Sigma_3^i)$ ...
The Dwyer number of a knot $K \subset \# S^2 \times S^1$

**Definition (Dwyer ‘75, reformulation by Cochran-Harvey ‘07)**

For a space $X$, $\Phi_n(X) \subset H_2(X)$ is the subgroup generated by homology classes which can be represented by maps of surfaces which are the first layer of an $n + 1$ half-grope.
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**Definition (K.)**

Let $K$ be a null-homologous knot in $\#^l S^2 \times S^1$. The Dwyer number of $K$ is

$$D(K) = \max \{ q \mid \frac{\text{H}_2(\#^l S^2 \times S^1 \setminus K)}{\Phi_q(\#^l S^2 \times S^1 \setminus K)} = 0 \}.$$
Why would this be the right definition?
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**Proposition (K.)**

If $K$ is a null-homologous knot in $\#^l S^2 \times S^1$ with $G = \pi_1(\#^l S^2 \times S^1 \setminus K, \ast)$, then $D(K) = q$ if and only if $G/G_k$ is isomorphic to $F/F_k$ for $k < q$ and $G/G_q$ is not isomorphic to $F/F_q$. 
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**Proposition (K.)**

*If* $K$ *is a null-homologous knot in* $\#^1 S^2 \times S^1$ *with* $G = \pi_1(\#^1 S^2 \times S^1 \setminus K, *)$, *then* $D(K) = q$ *if and only if* $G/G_k$ *is isomorphic to* $F/F_k$ *for* $k < q$ *and* $G/G_q$ *is not isomorphic to* $F/F_q$.

**Theorem (K.)**

*If* $K$ *is a null-homologous knot in* $\#^1 S^2 \times S^1$ *then* $D(K) \geq q$ *if and only if the longitude of* $K$ *lies in* $G_{q-1}$.
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Theorem (K.)

$D(K)$ is an invariant of concordance in $(\#^1 S^2 \times S^1) \times I$. 
Properties of $D(K)$.

Example

A knot $K \subset S^1 \times S^2$ with $D(K) = 4$
Properties of $D(K)$.

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- If every homology class in $\mathrm{H}_2(\#^lS^2 \times S^1 \setminus K)$ can be represented by a half-grope of arbitrary class, we say $D(K) = \infty$
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- If every homology class in $H_2(\#^1 S^2 \times S^1 \setminus K)$ can be represented by a half-grope of arbitrary class, we say $D(K) = \infty$.
- If $K$ is the unknot, $D(K) = \infty$.
- $3 \leq D(K) \leq \infty$. 
$D(K)$ behaves like first non-vanishing $\overline{\mu}_L(I)$.

**Theorem (K.)**

If $K$ is a null-homologous knot in $\#^1 S^2 \times S^1$ and $D(K) = q$, then the first non-vanishing Massey product in $H^1(\#^1 S^2 \times S^1 \setminus K, \ast)$ is weight $q$. 
$D(K)$ behaves like first non-vanishing $\overline{\mu}_L(I)$.

**Theorem (K.)**

*If $K$ is a null-homologous knot in $\#^1S^2 \times S^1$ and $D(K) = q$, then the first non-vanishing Massey product in $H^1(\#^1S^2 \times S^1 \setminus K, \ast)$ is weight $q$.***

**Theorem (K.)**

*There is an infinite family $\{M_i\}$ of null-homologous knots in $\#^1S^2 \times S^1$ which bound null-homologous disks in $\natural^1S^2 \times D^2$ and distinct in (stable) concordance.*
What does this mean?

For knots in \( K \subset \#^3 S^1 \times S^2 \),

\[
K_3 \subset \#^3 S^1 \times S^2 \text{ with } D(K) = 4
\]

concordance \( \implies \) slice in \( \#^l S^2 \times D^2 \)

slice in \( \#^l S^2 \times D^2 \) \( \implies \) concordance.
What linking data is preserved by knotification
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**Proposition (Ozsváth-Szabó ‘03)**

For every oriented $n$-component link $L \subset S^3$ we can construct a knot $\kappa(L) \subset \#^{n-1}S^1 \times S^2$ which is unique up to diffeomorphism of $\#^{n-1}S^1 \times S^2$ throwing one knot onto another. We call $\kappa(L)$ the knotification of $L$. 

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Matthew Hedden and I used the previous theorem in order to motivate the definition of a concordance group of knotified links.
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**Theorem (Hedden-K.)**

If a $L \subset S^3$ is an $n$-component link with first non-vanishing $\overline{\mu}_L(I)$ invariant weight $rn + 1$, then $D(\kappa(L)) \geq r + 1$. 
Future Goals

- Classify Dwyer number for knots and links in other 3-manifolds.
- Construct a version with rational coefficients to deal with knots in 3-manifolds with torsion.
- Identify higher order linking data within the link Floer complex (Gorsky-Liu-Moore '18 recovered the Sato-Levine invariant $\mu(1122)$ for 2-component links).
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Thank you!