

**MATH 4305, Fall 2014,  
Midterm 2, Practice: Solution**

Show all your work. You may use one side of a  $3 \times 5$  index card for formulars in this exam. Calculator is allowed. Please give yourself 50 minutes.

**Problem 1** Let  $\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ,  $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ , and  $W = \text{span}\{\mathbf{u}, \mathbf{v}\}$ .

a) Find the orthogonal projection of  $\mathbf{y}$  onto  $W$ .

**Solution** Note:  $\mathbf{u}$  is not orthogonal to  $\mathbf{v}$ . G-S process to an orthogonal basis of  $W$ .

Let  $\mathbf{w}_1 = \mathbf{u}$ ,  $\mathbf{w}_2 = \mathbf{v} - \frac{\mathbf{v} \bullet \mathbf{u}}{\mathbf{u} \bullet \mathbf{u}} \mathbf{u} = (0.5, -0.5, 1)^T$ .

We choose  $\{\mathbf{w}_1, \mathbf{w}'_2 = 2\mathbf{w}_2\}$  as the orthogonal basis for  $W$ .

$$\text{Proj}_W \mathbf{y} = \frac{\mathbf{y} \bullet \mathbf{w}_1}{\mathbf{w}_1 \bullet \mathbf{w}_1} \mathbf{w}_1 + \frac{\mathbf{y} \bullet \mathbf{w}'_2}{\mathbf{w}'_2 \bullet \mathbf{w}'_2} \mathbf{w}'_2 = \left(\frac{7}{3}, \frac{2}{3}, \frac{5}{3}\right)^T.$$

b) Find the distance between  $\mathbf{y}$  and  $W$ .

**Solution** Compute  $\|\mathbf{y} - \text{Proj}_W \mathbf{y}\| = \frac{4}{3}\sqrt{3}$ .

**Problem 2** Find the trigonometric function of the form  $f(t) = c_0 + c_1 \sin(t) + c_2 \cos(t)$  that best fits the data points  $(0, 0)$ ,  $(1, 1)$ ,  $(2, 2)$ ,  $(3, 3)$ , using least squares. Compute the least square error. (Remark: This is a problem for concept, find the formula, don't have to solve for exact solution. The test problem will be easier to solve.)

**Solution** We subject to solve the least square problem for  $A\mathbf{x} = \mathbf{b}$ , where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & \sin(1) & \cos(1) \\ 1 & \sin(2) & \cos(2) \\ 1 & \sin(3) & \cos(3) \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$

we thus solve the normal equation to obtain

$$\mathbf{c} = (A^T A)^{-1} A^T \mathbf{b}.$$

The least square error is  $\|\mathbf{b} - A\mathbf{c}\|$ .

**Problem 3** Find all possible values of  $a$  so that the columns of  $A$  given below are linearly dependent?

$$\begin{pmatrix} a & 2a & 0 & 0 \\ 0 & 0 & a-3 & 3(a-3) \\ 0 & -2a & 0 & 1 \\ 0 & 0 & a-2 & 2(a-2) \end{pmatrix}$$

**Solution:** Columns of  $A$  are linearly dependent if and only if  $\det(A) = 0$ . Compute the determinant using row operations and co-factor expansion, one has  $\det(A) = -2a^2(a-2)(a-3)$ . So,  $\det(A) = 0$  if  $a = 0$ , or  $a = 2$  or  $a = 3$ .

**Problem 4** (a) Prove that the set  $\mathbf{B} = \{1 + t^2, t + t^2, 1 + 2t + t^2\}$  is a basis for  $\mathbf{P}_2$ .

**Proof** Since  $\dim \mathbf{P}_2 = 3$  and  $\mathbf{B}$  has 3 vectors, it is sufficient to show that  $\mathbf{B}$  is linearly independent. Fix the standard basis  $S = \{1, t, t^2\}$ , we check the corresponding coordinate vectors are linearly independent. Let  $p_1(t) = 1 + t^2$ ,  $p_2(t) = t + t^2$ , and  $p_3(t) = 1 + 2t + t^2$ , we have

$$A = [[p_1]_S, [p_2]_S, [p_3]_S] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

We easily verify that  $A$  is invertible and thus proves that  $B$  is linearly independent and so is a basis for  $\mathbf{P}_2$ .

b) Find the matrix of the linear transformation  $T(f(t)) = f' - 3f$  from  $\mathbf{P}_2$  to  $\mathbf{P}_2$  with respect to the basis  $\mathbf{B}$  found in part (a).

**Solution:** The desired matrix  $M$  can be obtained by

$$AM = [[T(p_1)]_S, [T(p_2)]_S, [T(p_3)]_S] = D.$$

Since  $T(p_1) = -3 + 2t - 3t^2$ ,  $T(p_2) = 1 - t - 3t^2$  and  $T(p_3) = -1 - 4t - 3t^2$ , thus

$$M = A^{-1}D = A^{-1} \begin{pmatrix} -3 & 1 & -1 \\ 2 & -1 & -4 \\ -3 & -3 & -3 \end{pmatrix} = \begin{pmatrix} -4 & -\frac{1}{2} & 0 \\ 0 & -4 & -2 \\ 1 & \frac{3}{2} & -1 \end{pmatrix}.$$

**Problem 5.** Let  $A$  be the following matrix

$$\begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}$$

a) Find the  $QR$  factorization of  $A$ .

**Solution:** First of all, it is easy to verify that  $A$  has linearly independent columns. So,  $QR$  factorization is possible. Let the columns of  $A$  are  $\mathbf{x}_1$ ,  $\mathbf{x}_2$ , and  $\mathbf{x}_3$ . Then they form a basis for  $Col(A)$ .  $Q$  will be found by using G-S process with normalization on this basis. First of all, we find an orthogonal basis for  $Col(A)$  using G-S process:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1, 1)^T,$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1} \mathbf{v}_1 = (1, -1, -1, 1)^T,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{x}_3 \bullet \mathbf{v}_2}{\mathbf{v}_2 \bullet \mathbf{v}_2} \mathbf{v}_2 = (1, -1, 1, -1)^T.$$

Thus,  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is an orthogonal basis for  $Col(A)$ . This basis can be normalized into an orthonormal basis  $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$  with

$$\mathbf{u}_1 = \frac{1}{2}\mathbf{v}_1, \quad \mathbf{u}_2 = \frac{1}{2}\mathbf{v}_2, \quad \mathbf{u}_3 = \frac{1}{2}\mathbf{v}_3.$$

The matrix  $Q$  is now given by  $[\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3]$ , and  $R$  is given by  $Q^T A$ . The results are

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

$$R = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

b) Find the orthogonal projection of  $\mathbf{b} = (1, 2, 3, 4)^T$  onto  $Col(A)$ .

**Solution:**  $proj_{Col(A)} = QQ^T \mathbf{b} = \frac{1}{4} \begin{pmatrix} 3 & -1 & 1 & 1 \\ -1 & 3 & 1 & 1 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 3 \end{pmatrix} \mathbf{b} = (2, 3, 2, 3)^T.$

**Problem 6:** If  $A$  is an  $n \times n$  matrix, is it true that  $\det(AA^T) = \det(A^T A)$ ? Why?

**Solution:** Yes, both are  $(\det(A))^2$ .