MATH 4305, Fall 2014
Final Exam, Pratice

Show all your work. You may use both sides of an letter sized sheet paper (8.5 inch by 11 inch) for formulae in this exam. Please give yourself 120 minutes. Calculator is NOT allowed.

Problem 1 Please complete the on-line course survey. Thank you in advance.

Problem 2 Let $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

a) Find a basis for $Nul(A)$. Show your work.

b) What is the dimension of $Row(A)$? Mention a theorem to justify your answer.

c) Find the reduced SVD for A.

d) Find the optimal least-square solution to $A\mathbf{x} = \mathbf{b}$.

Problem 3. Let $Q(\mathbf{x}) = 4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 2x_1x_2 + 6x_1x_3 + 2x_1x_4 + 2x_2x_3 + 6x_2x_4 + 2x_3x_4$.

a) Find the matrix real symmetric matrix A for $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, compute its eigenvalues and verify if Q is positive definite.

b) Find an orthogonal matrix P such that the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $Q(\mathbf{x})$ into a quadratic form with no-cross product term. Give P and the new quadratic form.

c) Find the maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$. Find a unit vector $\mathbf{u}$ where the maximum is attained.

d) Find the minimum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$. Find a unit vector $\mathbf{v}$ where the minimum is attained.

Problem 4 Let $\mathbf{P}_3$ denote the vector space of polynomial of degree at most 3. Let $T : \mathbf{P}_3 \to \mathbf{P}_3$ be given by $T(p(t)) = p(t) - t$. For example, $T(1+2t+t^2) = 1 + t + t^2$. Is T a linear transformation? Explain your answer briefly in complete sentences.

Problem 5 A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

Problem 6. Compute e^{At} if

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Problem 7. Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation such that

$$T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \text{ and } T(\begin{bmatrix} -1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

a) Find the matrix of T relative to standard bases for $\mathbf{R}^2$.

b) Evaluate $T(\begin{bmatrix} 3 \\ 1 \end{bmatrix})$.

c) Find a basis $\mathbf{B}$ of $\mathbf{R}^2$ so that $\mathbf{B}$-matrix of T is diagonal if possible.

Problem 8. Let A be an $n \times n$ matrix such that $A^2 = 2A + 3I$. Show that A is invertible and 2 is not an eigenvalue. Find one eigenvalue of A, if -1 is not an eigenvalue.