MATH 4305, Fall 2014
Practice Final: Solutions

Problem 1 Please complete the on-line course survey. Thank you in advance.

Solutions: If you still did not do it, please do it right now.

Problem 2 Let $A = \begin{pmatrix} 4 & -2 \\ 2 & -1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

a) Find a basis for $Nul(A)$. Show your work.

Solution: Row reduce the augmented matrix into RREF

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So a bisis of $Nul(A)$ is $\{(\frac{1}{2}, 1)^T\}$.

b) What is the dimension of $Row(A)$? Mention a theorem to justify your answer.

Solution:By rank theorem, $rank(A)+dim\ Nul(A) = 2$, we know $dim\ Nul(A) = 1$ from part a), so $rank(A) = 1 = dim\ Row(A)$.

c) Find the reduced SVD for A.

Solution: Comput $A^T A = \begin{pmatrix} 20 & -10 \\ -10 & 5 \end{pmatrix}$. This martix has eigenvalues $\lambda_1 = 25$, and $\lambda_2 = 0$, so the singular value of A is $\sigma_1 = 5$. Compute the eigenspace of $A^T A$ corresponding to λ_1, normalize the eigenvector, one has $\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$. Then, compute $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$. Thus one finds the reduced SVD for A, $A = UDV^T$, where

$$U = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}, \ D = [5], \ V = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}.$$

d) Find the optimal least-square solution to $A\mathbf{x} = \mathbf{b}$.

Solution: the optimal least-square solution is $\mathbf{x}^+ = A^+\mathbf{b}$, where $A^+ = VD^{-1}U^T$. So,

$$\mathbf{x}^+ = \begin{pmatrix} \frac{8}{25} \\ \frac{-4}{25} \end{pmatrix}$$

Problem 3. Let $Q(\mathbf{x}) = 4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_4^2 + 2x_1x_2 + 6x_1x_3 + 2x_1x_4 + 2x_2x_3 + 6x_2x_4 + 2x_3x_4$.

a) Find the matrix of $Q(\mathbf{x})$, compute its eigenvalues and verify if Q is positive definite.

Solution: Matrix A is

$$A = \begin{pmatrix} 4 & 1 & 3 & 1 \\ 1 & 4 & 1 & 3 \\ 3 & 1 & 4 & 1 \\ 1 & 3 & 1 & 4 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = 9$, $\lambda_2 = 5$, $\lambda_3 = \lambda_4 = 1$, they all are positive, so Q is positive definite.

b) Find an orthogonal matrix P such that the change of variable $\mathbf{x} = P\mathbf{y}$ transforms $Q(\mathbf{x})$ into a quadratic form with no-cross product term. Give P and the new quadratic form.

solution: P will be formed by orthonormal eigenvectors of A. For λ_1, the unit eigenvector can be chosen as $\mathbf{v}_1 = (1/2, 1/2, 1/2, 1/2)^T$. For λ_2, the unit eigenvector can be chosen as $\mathbf{v}_2 = (-1/2, 1/2, -1/2, 1/2)^T$. For λ_3, the orthonormal eigenvectors can be chosen as $\mathbf{v}_3 = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)^T$ and

$\mathbf{v}_4 = (0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})^T$. So, $\mathbf{x} = P\mathbf{y}$ where $P = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$, and $Q(\mathbf{x}) =$

$9y_1^2 + 5y_2^2 + y_3^2 + y_4^2$.

c) Find the maximum value of $Q(\mathbf{x})$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$. Find a unit vector $\mathbf{u}$ where the maximum is attained.

solution: This maximum is attained at $\mathbf{u} = \mathbf{v}_1$, and the maximum is $\lambda_1 = 9$.

d) Find the minimum of $Q(\mathbf{x})$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$. Find a unit vector $\mathbf{v}$ where the minimum is attained.

solution: This minimum is attained at any unit vector that is a linear combination of $\mathbf{v}_3$ and $\mathbf{v}_4$. This minimum is 1.

Problem 4 Let $\mathbf{P}_3$ denote the vector space of polynomial of degree at most 3. Let $T : \mathbf{P}_3 \to \mathbf{P}_3$ be given by $T(p(t)) = p(t) - t$. For example, $T(1+2t+t^2) = 1 + t + t^2$. Is T a linear transformation? Explain your answer briefly in complete sentences.

Solution: $T(0) = -t$, so T does not map 0 to 0, and it is not linear.

Problem 5 (5 pt) A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

Solution: No, A is not diagonalizable. A is 3×3 and λ_1 and λ_2 are eigenvalues, we can assume that λ_1 is simple and λ_2 is double. It says, $dim(E_2) = 1$, it is less than algebraic multiplicity, so it is not diagonalizable.

Problem 6. Compute e^{At} if

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Solution: A has double eigenvalue 0 with one-dimensional eigenspace. One found eigenvector $\mathbf{v} = (-1, 1)^T$, generalized eigenvector $\mathbf{w} = (-1, 0)^T$. So, $A = PJP^{-1}$ with

$$P = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Notice that $J^2 = 0$, it is easy to compute:

$$e^{At} = Pe^{Jt}P^{-1} = \begin{pmatrix} t+1 & t \\ -t & 1-t \end{pmatrix}$$

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The purpose of this problem is to show you why Jordan form is useful. We will not test the Jordan form in the final.

Problem 7. Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation such that

$$T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \text{ and } T(\begin{bmatrix} -1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

a) Find the matrix of T relative to standard bases for $\mathbf{R}^2$.

Solution: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

b) Evaluate $T(\begin{bmatrix} 3 \\ 1 \end{bmatrix})$.

Solution: $(5,13)^T$.

c) Find a basis $\mathbf{B}$ of $\mathbf{R}^2$ so that $\mathbf{B}$-matrix of T is diagonal if possible.

Solution: $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the basis and $\mathbf{v}_1 = (\frac{\sqrt{33}-3}{6}, 1)^T$, $\mathbf{v}_2 = (-\frac{\sqrt{33}+3}{6}, 1)^T$.

Problem 8. Let A be an $n \times n$ matrix such that $A^2 = 2A + 3I$. Show that A is invertible and 2 is not an eigenvalue. Find one eigenvalue of A, if -1 is not an eigenvalue.

Solution: $A^2 - 2A = 3I$ implies that $det(A)det(A-2I) \neq 0$. Thus, $det(A) \neq 0$ implies that A is ivertible, and $det(A-2I) \neq 0$ means that 2 is not an eigenvalue of A.

We now make use of $A^2 - 2A - 3I = (A+I)(A-3I) = 0$, thus either $det(A+I) = 0$ or $det(A-3I) = 0$. -1 is not an eigenvalue, one concludes that $det(A-3I) = 0$ this means 3 is an eigenvalue of A.