MATH 4305, Fall 2015,
Midterm 2, Practice: Solution

Show all your work. You may use one side of a 3×5 index card for formulars in this exam. Calculator is NOT allowed. Please give yourself 50 minutes.

Problm 1 Let $\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $W = span\{\mathbf{u}, \mathbf{v}\}$.

a) Find the orthogonal projection of $\mathbf{y}$ onto W.

Solution Note: $\mathbf{u}$ is not orthogonal to $\mathbf{v}$. G-S process to an orthogonal basis of W.

Let $\mathbf{w}_1 = \mathbf{u}$, $\mathbf{w}_2 = \mathbf{v} - \frac{\mathbf{v} \bullet \mathbf{u}}{\mathbf{u} \bullet \mathbf{u}}\mathbf{u} = (0.5, -0.5, 1)^T$.

We choose $\{\mathbf{w}_1, \mathbf{w'}_2 = 2\mathbf{w}_2\}$ as the orthogonal basis for W.

$$Proj_W \mathbf{y} = \frac{\mathbf{y} \bullet \mathbf{w}_1}{\mathbf{w}_1 \bullet \mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{y} \bullet \mathbf{w'}_2}{\mathbf{w'}_2 \bullet \mathbf{w'}_2}\mathbf{w'}_2 = (\frac{7}{3}, \frac{2}{3}, \frac{5}{3})^T.$$

b) Find the distance between $\mathbf{y}$ and W.

Solution Compute $\|\mathbf{y} - Proj_W \mathbf{y}\| = \frac{4}{3}\sqrt{3}$.

Problem 2 Find the trigonometric function of the form $f(t) = c_0 + c_1 sin(t) + c_2 cos(t)$ that best fits the data points $(0,0)$, $(1,1)$, $(2,2)$, $(3,3)$, using least squares. Compute the least square error.(Remark: This is a problem for concept, find the formula, don't have to solve for exact solution. The test problem will be easier to solve.)

Solution We subject to solve the least square problem for $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & sin(1) & cos(1) \\ 1 & sin(2) & cos(2) \\ 1 & sin(3) & cos(3) \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$

we thus solve the normal equation to obtain

$$\mathbf{c} = (A^T A)^{-1} A^T \mathbf{b}.$$

The least square error is $\|\mathbf{b} - A\mathbf{c}\|$.

Problem 3 Find all possible values of a so that the columns of A given below are linearly dependent?

$$\begin{pmatrix} a & 2a & 0 & 0 \\ 0 & 0 & a-3 & 3(a-3) \\ 0 & -2a & 0 & 1 \\ 0 & 0 & a-2 & 2(a-2) \end{pmatrix}$$

Solution: Columns of A are linearly dependent if and only if $det(A) = 0$. Compute the determinant using row operations and co-factor expansion, one has $det(A) = -2a^2(a-2)(a-3)$. So, $det(A) = 0$ if $a = 0$, or $a = 2$ or $a = 3$.

Problem 4 (a) Prove that the set $\mathbf{B} = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbf{P}_2$.

Proof Since $dim\mathbf{P}_2 = 3$ and $\mathbf{B}$ has 3 vectors, it is sufficient to show that $\mathbf{B}$ is linearly independet. Fix the standard basis $S = \{1, t, t^2\}$, we check the corresponding coordinate vectors are linearly independent. Let $p_1(t) = 1+t^2$, $p_2(t) = t + t^2$, and $p_3(t) = 1 + 2t + t^2$, we have

$$A = [[p_1]_S, [p_2]_S, [p_3]_S] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

We easily verify that A is invertible and thus proves that B is linearly independent and so is a basis for $\mathbf{P}_2$.

b) Find the matrix of the linear transformation $T(f(t)) = f' - 3f$ from $\mathbf{P}_2$ to $\mathbf{P}_2$ with respect to the basis $\mathbf{B}$ found in part (a).

Solution: The desired matrix M can be obtained by

$$AM = [[T(p_1)]_S, [T(p_2)]_S], [T(p_3)]_S] = D.$$

Since $T(p_1) = -3+2t-3t^2$, $T(p_2) = 1-t-3t^2$ and $T(p_3) = -1-4t-3t^2$, thus

$$M = A^{-1}D = A^{-1}\begin{pmatrix} -3 & 1 & -1 \\ 2 & -1 & -4 \\ -3 & -3 & -3 \end{pmatrix} = \begin{pmatrix} -4 & -\frac{1}{2} & 0 \\ 0 & -4 & -2 \\ 1 & \frac{3}{2} & -1 \end{pmatrix}.$$

Problem 5. Let A be the following matrix

$$\begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}$$

a) Find the QR factorization of A.

Solution: First of all, it is easy to verify that A has linearly independent columns. So, QR factorization is possible. Let the columns of A are $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$. Then they form a basis for $Col(A)$. Q will be found by using G-S process with normalization on this basis. First of all, we find an orthogonal basis for $Col(A)$ using G-S process:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1,1,1,1)^T,$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1}\mathbf{v}_1 = (1,-1,-1,1)^T,$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \bullet \mathbf{v}_1}{\mathbf{v}_1 \bullet \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \bullet \mathbf{v}_2}{\mathbf{v}_2 \bullet \mathbf{v}_2}\mathbf{v}_2 = (1,-1,1,-1)^T.$$

Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $Col(A)$. This basis can be normalized into an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ with

$$\mathbf{u}_1 = \frac{1}{2}\mathbf{v}_1,\ \mathbf{u}_2 = \frac{1}{2}\mathbf{v}_2,\ \mathbf{u}_3 = \frac{1}{2}\mathbf{v}_3.$$

The matrix Q is now given by $[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$, and R is given by $Q^T A$. The results are

$$Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

$$R = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

b) Find the orthoganal projection of $\mathbf{b} = (1,2,3,4)^T$ onto $Col(A)$.

Solution: $proj_{Col(A)} = QQ^T\mathbf{b} = \frac{1}{4}\begin{pmatrix} 3 & -1 & 1 & 1 \\ -1 & 3 & 1 & 1 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 3 \end{pmatrix}\mathbf{b} = (2,3,2,3)^T.$

Problem 6: If A is an $n \times n$ matrix, is it true that $det(AA^T) = det(A^TA)$? Why?
Solution: Yes, both are $(det(A))^2$.