

**MATH 2551, Fall 2017**  
**Practice Midterm 1, Solutions**

**Problem 1.** Calculations.

(a)  $\frac{d}{dt}[(2t\mathbf{i} + \sqrt{t}\mathbf{j}) \bullet (t\mathbf{i} - 3\mathbf{j})]$

**solution:**  $4t - \frac{3}{2}t^{-1/2}$ .

(b)  $\frac{d}{dt}[(\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}) \times (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})]$

**solution:**  $(5\cos t - 4)\mathbf{i} + (5\sin t + 3)\mathbf{j} - (4\sin t + 3\cos t)\mathbf{k}$

(c)  $\frac{d}{dt}[e^{\cos 2t}\mathbf{i} + \ln(1 + t^2)\mathbf{j} + (1 - \cos t)\mathbf{k}]$

**solution:**  $-2\sin(2t)e^{\cos 2t}\mathbf{i} + \frac{2t}{1+t^2}\mathbf{j} + \sin t\mathbf{k}$

**Problem 2** A golf ball is hit at time  $t = 0$ . Its position vector as a function of time is given by

$$\mathbf{r}(t) = 2t\mathbf{i} + 3t\mathbf{j} + (-t^2 + 4t)\mathbf{k}.$$

Notice that at  $t = 0$  the ball is at the origin of the coordinate system. The  $xy$  plane represents the ground. At some time  $t_1 > 0$  the ball will return to the  $xy$  plane at some point  $P(a, b, 0)$ .

(a) Compute the velocity, the acceleration and the speed of the ball at an arbitrary time  $t$ .

**solution:**

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2\mathbf{i} + 3\mathbf{j} + (4 - 2t)\mathbf{k},$$

$$\mathbf{a}(t) = -2\mathbf{k},$$

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{13 + (4 - 2t)^2}.$$

(b) Find the time  $t_1 > 0$  and the coordinates of the point  $P$  where the ball hits the  $xy$  plane again.

**solution:**  $t_1 = 4$ .

(c) Set up a definite integral equal to the length of the arc of the trajectory from the origin to the point  $P$ . You do not have to evaluate the integral.

**solution:**  $\int_0^4 \sqrt{13 + (4 - 2t)^2} dt$ .

(d) Find the equation of the line tangent to the trajectory at  $P$ .

**solution:**  $\mathbf{R}(u) = 8\mathbf{i} + 12\mathbf{j} + u\mathbf{r}'(4) = (8 + 2u)\mathbf{i} + (12 + 3u)\mathbf{j} - 4u\mathbf{k}$ .

(e) Find the equation of the vertical plane containing the trajectory.

**solution:** The plane is through the origin and is vertical. The vector  $\mathbf{k}$  is in this plane. The other vector can be easily found by the origin and the point  $P$ , i.e.,  $8\mathbf{i} + 12\mathbf{j}$ , or simply  $2\mathbf{i} + 3\mathbf{j}$ . Thus the normal for the plane of the trajectory is

$$\mathbf{n} = (2\mathbf{i} + 3\mathbf{j}) \times \mathbf{k} = 3\mathbf{i} - 2\mathbf{j}.$$

The plane equation is thus

$$3x - 2y = 0.$$

(f) Find the curvature of the trajectory at P.

**solution:** (1)  $k = \frac{\|\mathbf{v} \times \mathbf{a}\|}{(ds/dt)^3}$ . At  $t_1 = 4$ ,

$$k = \frac{\sqrt{52}}{29^{\frac{3}{2}}}.$$

$$(2) k = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{\sqrt{52}}{29^{\frac{3}{2}}}.$$

**Problem 3** At each point  $P(x(t), y(t), z(t))$  of its motion, an object of mass  $m$  is subject to a force:

$$\mathbf{F}(t) = m\pi^2[4\cos(\pi t)\mathbf{i} + 3\sin(\pi t)\mathbf{j}].$$

Given that  $\mathbf{v}(0) = -3\pi\mathbf{j} + \mathbf{k}$ , and  $\mathbf{r}(0) = 3\mathbf{j}$ . find the following:

**solution:** Upon integration, we have

$$\mathbf{a}(t) = \pi^2[4\cos(\pi t)\mathbf{i} + 3\sin(\pi t)\mathbf{j}],$$

$$\mathbf{v}(t) = 4\pi\sin(\pi t)\mathbf{i} - 3\pi\cos(\pi t)\mathbf{j} + \mathbf{k}.$$

$$\mathbf{r}(t) = 4(1 - \cos(\pi t))\mathbf{i} + 3(1 - \sin(\pi t))\mathbf{j} + t\mathbf{k}.$$

(a) The velocity  $\mathbf{v}(1)$ .

**solution:**  $\mathbf{v}(1) = 3\pi\mathbf{j} + \mathbf{k}$ .

(b) The speed  $v(1)$ .

**solution:**  $v(1) = \sqrt{9\pi^2 + 1}$ .

(c) The momentum  $\mathbf{p}(1)$ .

**solution:**  $\mathbf{p}(1) = m\mathbf{v}(1) = 3m\pi\mathbf{j} + m\mathbf{k}$ .

(d) The angular momentum  $\mathbf{L}(1)$ .

**solution:**  $\mathbf{L}(1) = \mathbf{r}(1) \times \mathbf{p}(1) = 3m(1 - \pi)\mathbf{i} - 8m\mathbf{j} + 24m\pi\mathbf{k}$ .

(e) The torque  $\boldsymbol{\tau}(1)$ .

**solution:**  $\boldsymbol{\tau}(1) = \mathbf{r}(1) \times \mathbf{F}(1) = -4m\pi^2\mathbf{j} + 12m\pi^2\mathbf{k}$ .

(f) The position  $\mathbf{r}(1)$ .

**solution:**  $\mathbf{r}(1) = 8\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ .

(g) The osculating plane equation at  $\mathbf{r}(1)$ .

**solution:** We know that both  $\mathbf{v}(1) = 3\pi\mathbf{j} + \mathbf{k}$  and  $\mathbf{a}(1) = -4\pi^2\mathbf{i}$  are in this osculating plane, therefore the normal vector can be chosen any nonzero multiple of

$$\mathbf{v}(1) \times \mathbf{a}(1) = (3\pi\mathbf{j} + \mathbf{k}) \times (-4\pi^2\mathbf{i}) = -4\pi^2\mathbf{j} + 12\pi^3\mathbf{k}.$$

Here, we choos  $\mathbf{n} = -\mathbf{j} + 3\pi\mathbf{k}$ . Therefore, the plane equation is

$$-(y - 3) + 3\pi(z - 1) = 0.$$

(h) The tengential and normal components of accelaration  $\mathbf{a}(1)$ .

**solution:**

$$a_T(1) = \frac{\mathbf{v}(1) \cdot \mathbf{a}(1)}{v(1)} = 0,$$

$$a_N(1) = \frac{\|\mathbf{v}(1) \times \mathbf{a}(1)\|}{v(1)} = 4\pi^2.$$

Remark: in this problem, from  $a_T(1) = 0$ , you see that  $\mathbf{a}(1)$  has only normal component,therefore,  $a_N(1) = \|\mathbf{a}(1)\| = 4\pi^2$