MATH 2551, Fall 2018
Practice Midterm 1, Solutions

Problem 1. Calculations.

(a) $\frac{d}{dt}[(2t\mathbf{i} + \sqrt{t}\mathbf{j}) \bullet (t\mathbf{i} - 3\mathbf{j})]$

solution: $4t - \frac{3}{2}t^{-1/2}$.

(b) $\frac{d}{dt}[(cost\mathbf{i} + sint\mathbf{j} + t\mathbf{k}) \times (3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})]$

solution: $(5cost - 4)\mathbf{i} + (5sint + 3)\mathbf{j} - (4sint + 3cost)\mathbf{k}$

(c) $\frac{d}{dt}[e^{cos2t}\mathbf{i} + ln(1+t^2)\mathbf{j} + (1 - cost)\mathbf{k}]$

solution: $-2sin(2t)e^{cos2t}\mathbf{i} + \frac{2t}{1+t^2}\mathbf{j} + sint\mathbf{k}$

Problem 2 A golf ball is hit at time $t = 0$. Its position vector as a function of time is given by

$$\mathbf{r}(t) = 2t\mathbf{i} + 3t\mathbf{j} + (-t^2 + 4t)\mathbf{k}.$$

Notice that at $t = 0$ the ball is at the origin of the coordinate system. The xy plane represents the ground. At some time $t_1 > 0$ the ball will return to the xy plane at some point $P(a, b, 0)$.

(a) Compute the velocity, the accelaration and the speed of the ball at an arbitrary time t.

solution:

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2\mathbf{i} + 3\mathbf{j} + (4 - 2t)\mathbf{k},$$

$$\mathbf{a}(t) = -2\mathbf{k},$$

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{13 + (4 - 2t)^2}.$$

(b) Find the time $t_1 > 0$ and the coordinates of the point P where the ball hits the xy plane again.

solution: $t_1 = 4$.

(c) Set up a definite integral equal to the length of the arc of the trajectory from the origin to the point P. You do not have to evaluate the integral.

solution: $\int_0^4 \sqrt{13 + (4 - 2t)^2}\, dt.$

(d) Find the equation of the line tangent to the trajectory at P.

solution: $\mathbf{R}(u) = 8\mathbf{i} + 12\mathbf{j} + u\mathbf{r}'(4) = (8 + 2u)\mathbf{i} + (12 + 3u)\mathbf{j} - 4u\mathbf{k}.$

(e) Find the equation of the vertical plane containing the trajectory.

solution: The plane is through the origin and is vertical. The vector $\mathbf{k}$ is in this plane. The other vector can be easily found by the origin and the point P, i.e., $8\mathbf{i} + 12\mathbf{j}$, or simply $2\mathbf{i} + 3\mathbf{j}$. Thus the normal for the plane of the trajectory is

$$\mathbf{n} = (2\mathbf{i} + 3\mathbf{j}) \times \mathbf{k} = 3\mathbf{i} - 2\mathbf{j}.$$

The plane equation is thus

$3x - 2y = 0.$

(f) Find the curvature of the trajectory at P.

solution: (1)$k = \frac{\|\mathbf{v}\times\mathbf{a}\|}{(ds/dt)^3}$. At $t_1 = 4$,

$$k = \frac{\sqrt{52}}{29^{\frac{3}{2}}}.$$

(2) $k = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{\sqrt{52}}{29^{\frac{3}{2}}}$.

Problem 3 At each point $P(x(t), y(t), z(t))$ of its motion, an object of mass m is subject to a force:

$$\mathbf{F}(t) = m\pi^2[4cos(\pi t)\mathbf{i} + 3sin(\pi t)\mathbf{j}].$$

Given that $\mathbf{v}(0) = -3\pi\mathbf{j} + \mathbf{k}$, and $\mathbf{r}(0) = 3\mathbf{j}$. find the following:

solution: Upon integration, we have

$\mathbf{a}(t) = \pi^2[4cos(\pi t)\mathbf{i} + 3sin(\pi t)\mathbf{j}],$

$\mathbf{v}(t) = 4\pi sin(\pi t)\mathbf{i} - 3\pi cos(\pi t)\mathbf{j} + \mathbf{k}.$

$\mathbf{r}(t) = 4(1 - cos(\pi t))\mathbf{i} + 3(1 - sin(\pi t)\mathbf{j} + t\mathbf{k}.$

(a) The velocity $\mathbf{v}(1)$.

solution: $\mathbf{v}(1) = 3\pi\mathbf{j} + \mathbf{k}$.

(b) The speed $v(1)$.

solution: $v(1) = \sqrt{9\pi^2 + 1}$.

(c) The momentum $\mathbf{p}(1)$.

solution: $\mathbf{p}(1) = m\mathbf{v}(1) = 3m\pi\mathbf{j} + m\mathbf{k}$.

(d) The angular momentum $\mathbf{L}(1)$.

solution: $\mathbf{L}(1) = \mathbf{r}(1) \times \mathbf{p}(1) = 3m(1 - \pi)\mathbf{i} - 8m\mathbf{j} + 24m\pi\mathbf{k}$.

(e) The torque $\tau(1)$.

solution: $\tau(1) = \mathbf{r}(1) \times \mathbf{F}(1) = -4m\pi^2\mathbf{j} + 12m\pi^2\mathbf{k}$.

(f) The position $\mathbf{r}(1)$.

solution: $\mathbf{r}(1) = 8\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

(g) The osculating plane equation at $\mathbf{r}(1)$.

solution: We know that both $\mathbf{v}(1) = 3\pi\mathbf{j} + \mathbf{k}$ and $\mathbf{a}(1) = -4\pi^2\mathbf{i}$ are in this osculating plane, therefore the normal vector can be choosen any nonzero multiple of

$$\mathbf{v}(1) \times \mathbf{a}(1) = (3\pi\mathbf{j} + \mathbf{k}) \times (-4\pi^2\mathbf{i}) = -4\pi^2\mathbf{j} + 12\pi^3\mathbf{k}.$$

Here, we choos $\mathbf{n} = -\mathbf{j} + 3\pi\mathbf{k}$. Therefore, the plane equation is

$$-(y-3) + 3\pi(z-1) = 0.$$

(h) The tengential and normal components of accelaration $\mathbf{a}(1)$.

solution:

$$a_T(1) = \frac{\mathbf{v}(1) \cdot \mathbf{a}(1)}{v(1)} = 0,$$

$$a_N(1) = \frac{\|\mathbf{v}(1) \times \mathbf{a}(1)\|}{v(1)} = 4\pi^2.$$

Remark: in this problem, from $a_T(1) = 0$, you see that $\mathbf{a}(1)$ has only normal component,therefore, $a_N(1) = \|\mathbf{a}(1)\| = 4\pi^2$