MATH 2403, Section M1, M2 and M3, Spring 2013
Practice Exam I: Solutions

Problem 1. Solve each of the following, explicitly if possible and implicitly otherwise.

(a) $2xy' + y = 10\sqrt{x}$.

Solutions: Standard form

$$y' + \frac{1}{2x}y = 5x^{-1/2}.$$

The integrating factor is

$$\mu(x) = \sqrt{x}.$$

The solution is

$$y(x) = 5\sqrt{x} + cx^{-1/2}.$$

(b) $\frac{dy}{dx} = 2xy^2 + 3x^2y^2$

Solutions: The right hand side is $(2x + 3x^2)y^2$, so it is separable. $y = 0$ is a constant solution. For $y \neq 0$, separation of variables gives

$$y = -\frac{1}{x^2 + x^3 + C}.$$

(c) $6xy^3 + 2y^4 + (9x^2y^2 + 8xy^3)y' = 0$

Solutions: Standard form is

$$Mdx + Ndy = 0$$

where $M(x,y) = 6xy^3 + 2y^4$, and $N(x,y) = 9x^2y^2 + 8xy^3$. Since $M_y = N_x = 18xy^2 + 8y^3$, we know the equation is exact.

$$F = \int Mdx + g(y) = 3x^2y^3 + 2xy^4 + g(y).$$

$$F_y = 9x^2y^2 + 8xy^3 + g'(y) = N(x,y)$$

implies $g'(y) = 0$, so $g(y) = C$. Therefore, the solution is

$$F(x,y) = 3x^2y^3 + 2xy^4 = C.$$

Problem 2. Consider the initial value problem

$dy/dx = x^2 - y^2,\ y(0) = 1.$

(a) How many solutions are there solving this initial value problem? How do you know? Verify your answer using the proper theorem.

Solutions: $x_0 = 0$, $y_0 = 1$, and $f(x,y) = x^2 - y^2$. $f_y = -2y$. We see both f and f_y are continuous everywhere. By the Theorem of Existence and Uniqueness, we know this problem has a unique solution.

(b) Approximate $y(0.2)$ using Euler's method with step size 0.1. Show your steps.

Solutions: $x_0 = 0$, $y_0 = 1$ and $h = 0.1$. $x_1 = 0.1$, $y_1 = y_0 + f(x_0, y_0)h = 0.9$. $x_2 = 0.2$, $y_2 = y_1 + f(x_1, y_1)h = 0.82$ So the approximate value of $y(0.2)$ is 0.82.

Problem 3 A tank with 200 liters volume is full of pure water initially. A mixture containing a concentration of 6 gram/liter of salt enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression for the amount of salt in the tank at time t. How much salt is in the tank after 120 minute? What is the limiting amount of salt in the tank as $t \to \infty$?

Solution: Let $q(t)$ be the quantity of salt in the tank. The initial value problem is

$$q' = 12 - \frac{q}{100}, \ q(0) = 0.$$

Clearly, $q = 1200$ is not the solution. Solve the problem by separation of variables, we have

$$q(t) = 1200 + Ae^{-\frac{1}{100}t}.$$

By initial condition, one has $A = -1200$, and so

$$q(t) = 1200(1 - e^{-\frac{1}{100}t}).$$

When $t = 120$, $q(120) = 1200(1 - e^{-1.2})(g)$, or approximately 838.57 grams. The limiting value is 1200 grams.

Problem 4 (30 points) A chemical reaction can be modeled by the following equation

$dx/dt = \alpha(p - x)(q - x)$. Where α, and p and q are positive constants.

(a) If $p < q$, find the equilibrium solutions for this equation. Indicate the stable and unstable equilibrium on the phase diagram.

Solutions: $f(x) = \alpha(p - x)(q - x) = 0$ implies that $x_1 = p$ and $x_2 = q$ are two equilibrium solutions. The phase diagram will contain the following information:

If $x < p$, $f(x) > 0$; if $x \in (p, q)$, $f(x) < 0$; if $x > q$, $f(x) > 0$. Therefore, $x_1 = p$ is stable, $x_2 = q$ is not.

(b) If $x(0) = \frac{p+q}{2}$, $p < q$, determine the limiting value of $x(t)$ as $t \to +\infty$ without solving the differential equation.

Solutions: By part (a) and $\frac{p+q}{2} \in (p, q)$, we know the limiting value of $x(t)$ is p.

(c) If $x(0) = \frac{1}{2}$, $p = q = 1$, solve the initial value problem for this equation and determine the limiting value of $x(t)$ as $t \to +\infty$.

Solutions: $p = q = 1$, the problem reduced to

$$x' = \alpha(x-1)^2, \; x(0) = \frac{1}{2}.$$

$x = 1$ is not the solution. By separation of variables, one has

$$x(t) = 1 - \frac{1}{\alpha t + C},$$

where C is determined by initial condition and $C = 2$. So

$$x(t) = 1 - \frac{1}{\alpha t + 2},$$

and the limiting value is 1.