# MATH 2403, Spring 2013
## Practice II, Solutions

**Problem 1** Consider the system of differential equations

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = \frac{x-ty}{1-t}. \end{cases} \tag{1}$$

(a) What are the restrictions on $t_0$, $x_0$ and $y_0$ in order that this system of equations has a unique solution for which $x(t_0) = x_0$ and $y(t_0) = y_0$.

**Solution:** According to the existence and uniqueness theorem for the initial value problem for the linear homogeneous system, we know that the coeeficients need to be continuous. Therefore, we find the restriction

$$t_0 \neq 1.$$

(b) Find a solution $(x(t), y(t))$ of (1) for which $y(t) = 1$ is constant.

**Solution** Substitute $y = 1$ into the system, one finds

$$x' = 1,\ 0 = y' = \frac{x-t}{1-t},$$

which give $x = t$. Therefore, the desired solution is $x = t$ and $y = 1$.

(c) Verify that $(x, y) = (e^t, e^t)$ is another solution of (1).

**Solution:** Substitute and check. Yes.

(d) Find a solution which satisfies the initial conditions $x(0) = 1$ and $y(0) = 0$.

**Solution:** For this problem, we first try to see if two solutions given in (b) and (c) are linearly independent. For this purpose, we compute the Wronskian:

$$W(t) = det \begin{pmatrix} t & e^t \\ 1 & e^t \end{pmatrix} = (t-1)e^t \neq 0, \ if \ t \neq 1.$$

Therefore the general solution is given as

$$x(t) = c_1 t + c_2 e^t, \ y(t) = c_1 + c_2 e^t.$$

We now use the initial condition to determine $c_1$ and $c_2$. By substitution, we find $c_1 = -1$ and $c_2 = 1$.

**Problem 2**. Consider the predator-prey system

$$\begin{cases} \frac{dx}{dt} = 5x - xy \\ \frac{dy}{dt} = -2y + xy \end{cases}$$

(a) Find all critical points for this system.

**Solution:** Solve the system

$$\begin{cases} F(x,y) = 5x - xy = 0 \\ G(x,y) = -2y + xy = 0 \end{cases}$$

It gives the three critical points: $P_1(0,0)$ and $P_2(2,5)$.

(b) There is a critical point in first quadrant (if not, check your answer in (a)), find the corresponding linearization of your system near this critical point.

**Solution:** The critical point is $P_3(2,5)$. To linearlize the system near this point we compute the Jocobian

$$J(2,3) = \begin{pmatrix} 0 & -2 \\ 5 & 0 \end{pmatrix}$$

and the linearized system in variables $(u,v)$ is

$$\begin{cases} \frac{du}{dt} = -2v \\ \frac{dv}{dt} = 5u \end{cases}$$

(c) Sketch the phase portraits for the linear system you found in part (b), indicating the type and the stability of the critical point for this linear system.

**Solution:** We first compute the eigenvalues and found two complex ones $\lambda_{1,2} = \pm i\sqrt{10}$. Therefore, the origin is a stable center. At $(0, 1)$, the right hand side reads $(-2, 0)$, means the trajectory is going counterclockwise. You should now able to sketch its phase portrait.

(d) By integrating $\frac{dy}{dx}$, find the equation of trajectory curves for the nonlinear system in the phase plane.

**Solution:** By chain rule, we reach the equation

$$\frac{dy}{dx} = \frac{-2y + xy}{5x - xy}.$$

This is a separable equation.

$$\int \frac{5-y}{y}\, dy = \int \frac{x-2}{x}\, dx + C$$

this implies that

$$5\ln(y) - y = x - 2\ln(x) + C$$

**Problem 3** Consider the system of equations

$$\begin{cases} \frac{dx}{dt} = -8x + 5y, \\ \frac{dy}{dt} = -10x + 7y, \end{cases} \tag{2}$$

(a) Find the general solution of (2).

**Solution:** The eigenvalue are $\lambda_1 = -3$ and $\lambda_2 = 2$. For $\lambda_1 = -3$, we have the eigenvector $\mathbf{v}_1 = (1, 1)^T$; for $\lambda_2 = 2$, we have the eigenvector $\mathbf{v}_2 = (\frac{1}{2}, 1)^T$. Therefore, the solution is

$$c_1\mathbf{v}_1 e^{-3t} + c_2\mathbf{v}_2 e^{2t}.$$

(b) Sketch the solutions from (a) in the $(x, y)$ plane, state the type and stability of the origin, indicate clearly in which direction $t$ increases and the behavior for large $t$.

**Solution:** The sketch is omitted. Since $\lambda_1 < 0 < \lambda_2$, the origin is the saddle point, and so it is unstable. For increasing $t$, the solution will go to infinity except for the line $(y = x)$ in the direction of $\mathbf{v}_1$ throught the origin, along which, the solution decays to the origin.

(c) From your sketch in part (b), what can you say about the long-term behavior of the solution with initial condition $x(0) = 5$, $y(0) = 11$?

**Solution:** Since the initial point $(5, 11)$ is not on the line $y = x$, the solution will tend to infinity as $t$ goes to positive infinity.

(d) Let $A$ be the coefficient matrix in system (2). Compute $e^A$.

**Solution:** There are two ways to compute $e^A$: the fundamental matrix, and the matrix technique. Here we show the matrix technique. Clearly, let $P = (\mathbf{v}_1, \mathbf{v}_2)$ and $D = diag\{\lambda_1, \lambda_2\}$, one has $A = PDP^{-1}$, and

$$e^A = Pe^D P^{-1}, \quad for\ e^D = diag\{e^{\lambda_1}, e^{\lambda_2}\}.$$

Therefore, one finds

$$e^A = \begin{pmatrix} 2e^{-3} - e^2 & -e^{-3} + e^2 \\ 2e^{-3} - 2e^2 & -e^{-3} + 2e^2 \end{pmatrix}.$$