MATH 2403, Sring 2013
Final exam, Practice

Guideline: Please read the following carefully.

Remember to show all your work; including all intermediate steps and also explain in words how you are solving a problem. Partial credits are available for most problems. You may use one side of an A4 paper of formulas. A same table of the Laplace transform in the textbook will be provided in the test. Calculator is NOT allowed in this exam. You have 2 hours.

Problem 1 Newton's law of cooling says that the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature A is proportional to the difference $A - T$. A 4-lb roast, initially at 50 F, is placed in a 375 F oven at 5:00 pm. After 75 minutes it is found that the temperature $T(t)$ of the roast is 125 F. When will the roast be 150 F?

Problem 2. Solve the following problems.

(a) $y''' + 4y' = 0$, $y(0) = 0$, $y'(0) = 1$ and $y''(0) = 0$.

(b) $y'' + 3y' + 2y = 3x$.

(c) $y'' + 4y' + 4y = 2e^{-2x}$.

Problem 3. Consider the differential equation $4t^2y'' - ty' + y = 0$ (which we don't know how to solve).

(a) Show that $y_1(t) = t^{\frac{1}{4}}$ and $y_2(t) = t$ are solutions to this differential eqution (for $t > 0$).

(b) Write the general solution to the differential equation. What is true about y_1 and y_2 that allows you to do this? How do you know?

(c) Find the particular solution for the differential equatin if $y(1) = 3$ and $y'(1) = 2$.

(d) Find the general solution to the equation $4t^2y'' - ty' + y = 5t$

Problem 4 Consider the following system

$$\begin{cases} \dfrac{dx}{dt} = xy - 1 \\ \dfrac{dy}{dt} = x - y \end{cases}$$

(a) Find all critical points for this system.

Solutions: $P_1(1, 1)$ and $P_2(-1, -1)$.

(b) There is a critical point in 3rd quadrant, find the corresponding linearization of your system near this critical point.

(c) Find the general solution of this linear system.

(d) Sketch the phase portrait for the linear system you found in part (b), indicating the type and the stability of the critical point for this linear system.

Problem 5 Consider the system of equations

$$\begin{cases} \frac{dx}{dt} = -x - 4y, \\ \frac{dy}{dt} = x - y, \end{cases} \tag{1}$$

(a) Find the general solution of (1).

(b) Find the particular solution for system (1) with the initial condition $x(0) = 2$ and $y(0) = -1$.

(c) Sketch the the solution curves in (b) on the phase plane. Indicate the direction for t increase and the limiting behavior of the solution as t goes to $+\infty$.

Problem 6 (a) Find the Laplace transform of $f(t) = 4\delta(t-2) + t^3 e^{-3t} + g(t)$ where $g(t) = t$ if $t < 3$ and $g(t) = 3$ if $t > 3$.

(b) Use Laplace transform to solve the initial value problem

$$x'' + 3x' + 2x = t + 1;\ x(0) = 0,\ x'(0) = 2.$$

Problem 7 The charge on a capacitor, $Q(t)$, in a simple LRC circuit is given by $LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$, where L, R, and C are the inductance, resistance and capacitance of the inductor, resistor and capacitor in the system. Suppose that $R = 1k\Omega$, $C = 20\mu F$, $L = 0.35h$, and that the system is forced with an applied charge $E(t) = Asin(\omega t)$.

(a) For what ω, if any, will this system exhibit resonance? Explain.

(b) Now suppose $R = 0$ (the resistor is removed). If $\omega = 120\pi$ and $A = 117$, find the charge $Q(t)$ if $Q(0) = Q'(0) = 0$. Be sure that it is clear why you proceed as you do and how you arrive at your answer. Will your solution exhibit resonance, beats or neither? Why?